Math 43900 Problem Solving
Fall 2016
Lecture 2 Exercises

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These problems are taken from the textbook, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

Proof by contradiction

1. Every point of the plane is colored in one of two colors. Show that there exists an equilateral triangle all of whose vertices are colored with the same color.

   Proof. Did in class $\square$

2. Show that the equation $x^2 + x = 3^y$ has no integer solutions. [Hint: parity.]

   Proof. LHS always even, RHS always odd $\square$

3. Show that there exists no function $f : \mathbb{Z} \to \{a, b, c\}$ such that $f(x) \neq f(y)$ whenever $|x - y| \in \{2, 3, 5\}$.

   Proof. AG 6 $\square$

4. Every point in 3d space is colored red, green or blue. Show that at least one color has the following property: for every $a > 0$ there exist two points of this color at distance a from each other.

   Proof. AG 4 $\square$

Mathematical induction

Induction where you know what you need to show

1. Show that a $2^n \times 2^n$ board with one unit square removed can be tiled with corner tiles (a corner tiles has 3 unit square in an L shape).

   Proof. AG 18 $\square$

2. For an integer n define $f(n)$ by the following rules: $f(1) = 1$, $f(2n) = f(n)$ and $f(2n + 1) = f(n) + 1$. Show that $f(n)$ is the number of 1s in the binary representation of n.

   Proof. $2n$ adds a 0 to n written in base 2 while $2n + 1$ adds a 1 $\square$

3. Prove for all positive numbers the identity

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2n-1}-\frac{1}{2n}$$

Proof. AG 11 □

4. Let F_n be the n-th Fibonacci number. Show that $F_{n+1}^2+F_n^2=F_{2n+1}$ and $F_{n+1}^2+2F_nF_{n+1}=F_{2n+2}$.

Proof. Show BOTH by induction at the same time. See AG 24 □

5. A finite number of lines partitions the plane into regions. Show that these regions can be colored in two colors such that any two regions sharing an edge will have different colors.

Proof. When adding a new line, on one side of the line flip all colors □

6. A sequence $(x_n)_{n\geq 0}$ is defined by the recurrence relation $x_n = ax_{n-1}+bx_{n-2}$ for $n\geq 2$. Look at the quadratic equation $X^2-aX-b=0$ with *distinct* roots α and β. Show that you can find two numbers u and v such that $x_n = u\alpha^n+v\beta^n$ for all $n\geq 0$.

   (a) For the Fibonacci numbers $F_0=F_1=1$, $F_n=F_{n-1}+F_{n-2}$ this is $F_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$.

   How many digits does $F_{1000,000}$ have?

   (b) What about the sequence (x_n) with $x_n=2x_{n-1}-x_{n-2}$ with $x_1=1$, $x_2=2$? What is x_n?

Proof. The general formula I did in class. Part 1 is an application, see the wiki page for Fibonacci numbers. For part 2, $x_n=n$ by induction. □

7. Show that

$$2!4!\cdots(2n)!\geq((n+1)!)^n$$

Induction where you need to figure out what you want to prove

If you don't know what precise statement to prove by induction, you should try some small cases to guess the statement you'd like to prove.

1. Find a formula for the sum of the first n odd numbers.

Proof. n^2 □

2. Show that for all positive integers n

$$1+\frac{1}{2^3}+\cdots+\frac{1}{n^3}<\frac{3}{2}$$

[As it stands this looks hard to tackle by induction. Amusingly, the slightly harder inequality where you replace $<\frac{3}{2}$ with $\frac{3}{2}-\frac{1}{n}$ can be done with induction. What is the base case?]

Proof. AG 16 □

3. Find a formula for x_n knowing that $x_1=\frac{5}{2}$ and $x_{n+1}=x_n^2-2$ for all $n\geq 1$.

Proof. Part of Putnam 2014 A3 □

The pigeonhole principle

Geometrically the pigeonhole principle states that if you have a number of subsets of a bigger geometric set with total length/area/volume larger than the length/area/volume of the bigger set then at least two of the smaller subsets must intersect.

1. Prove that there are two non-bald people in the US with the same number of hairs on their heads.

   Proof. There are more people in the US than hairs on a head. □

2. Show that at any party there are two people who know exactly the same number of people at the party.

3. Consider integers $1 \leq a_1 < a_2 < \ldots < a_{50} < 100$. Show that $a_i + a_j = 99$ for some i and j.

   Proof. AG 33 □

4. Show that in any group of 6 people you can find 3 who know each other or 3 who are strangers to each other.

   Proof. Classical problem. See wiki on theorem on friends and strangers. □

5. Show that every convex polyhedron has two sides with the same number of edges. Can you give an example of a convex polyhedron with no 3 faces with the same number of edges?

   Proof. AG 46 □

6. If α is irrational and $\varepsilon > 0$ show that there exist two integers m and n such that $|m\alpha - n| < \varepsilon$. Deduce that for every sequence of digits $\overline{a_1 \ldots a_k}$ in base 10 some power of 2, when written in base 10, starts with this sequence of digits.

   Proof. Classical Diophantine approximation problem. See wiki for Dirichlet approximation. □

7. Inside a circle of radius 4 are 45 points. Show that you can find two of these points at most $\sqrt{2}$ apart. [Hint: Draw circles around each point.]

   Proof. Variant of AG 44, but using circles of radius $\sqrt{2}/2$ around each point instead of squares. □