Math 43900 Problem Solving
Fall 2016
Lecture 3 Exercises

Andrei Jorza

These problems are taken from the textbook, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

Stats

I ran some stats on Putnam problems and saw that about a quarter of the problems are combinatorics, a quarter are number theory. The next batch is geometry, algebra, analysis and inequalities with about 15% each. The third batch, with about 10% each, consists of functions, sequences, polynomials, matrices. Other nontrivial categories are derivatives, integrals, limits, equations, series, probabilities, abstract algebra.

Since most problems belong to more than one category, and it's sometimes not clear what category a problem belongs to, these stats are purely qualitative.

Mathematical induction

Induction where you know what you need to show

1. A sequence $(x_n)_{n\geq 0}$ is defined by the recurrence relation $x_n = ax_{n-1} + bx_{n-2}$ for $n \geq 2$. Look at the quadratic equation $X^2 - aX - b = 0$ with *distinct* roots α and β. Show that you can find two numbers u and v such that $x_n = u\alpha^n + v\beta^n$ for all $n \geq 0$.

   (a) For the Fibonacci numbers $F_0 = F_1 = 1$, $F_n = F_{n-1}+F_{n-2}$ this is $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$.

   How many digits does $F_{1000,000}$ have?

   (b) What about the sequence (x_n) with $x_n = 2x_{n-1} - x_{n-2}$ with $x_1 = 1$, $x_2 = 2$? Show that $x_n = n$ by induction. (The discrepancy is due to Jordan canonical forms for matrices, we'll see later.)

   Proof. The general formula I did in class. Part 1 is an application, see the wiki page for Fibonacci numbers. For part 2, $x_n = n$ by induction. □

2. (Putnam 2015) Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find a closed formula for a_n. (On the Putnam the question was find an odd prime factor of a_{2015}. We'll see this next week when we do polynomials.)

   Proof. $a_n = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2$ from the previous exercise. □

3. Show by induction that for $n \geq 1$, 2^{n+3} divides $3^{2^n} - 2^{n+2} - 1$. (I used this to solve a problem on the IMO 2000.)

Induction where you need to figure out what you want to prove

If you don't know what precise statement to prove by induction, you should try some small cases to guess the statement you'd like to prove.

1. Find a formula for the sum of the first n odd numbers.

   Proof. n^2 $\square$

2. Show that for all positive integers n

$$1 + \frac{1}{2^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}$$

   [As it stands this looks hard to tackle by induction. Amusingly, the slightly harder inequality where you replace $< \frac{3}{2}$ with $< \frac{3}{2} - \frac{1}{n}$ can be done with induction. What is the base case?]

   Proof. AG 16 $\square$

3. Find a formula for x_n knowing that $x_1 = \frac{5}{2}$ and $x_{n+1} = x_n^2 - 2$ for all $n \geq 1$.

   Proof. Part of Putnam 2014 A3 $\square$

4. Find a closed formula for $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^n$.

   Proof. $\begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}$ $\square$

5. (*) Find a formula for x_n knowing that $x_0 = x_1 = 1$ and for $n \geq 0$, $x_{n+2} = x_{n+1} + 2x_n + 2n - 1$.

   Proof. $x_n = 2^n - n$ $\square$

The pigeonhole principle

Geometrically the pigeonhole principle states that if you have a number of subsets of a bigger geometric set with total length/area/volume larger than the length/area/volume of the bigger set then at least two of the smaller subsets must intersect.

1. Prove that there are two non-bald people in the US with the same number of hairs on their heads.

   Proof. There are more people in the US than hairs on a head. $\square$

2. Show that at any party there are two people who know exactly the same number of people at the party.

3. Consider integers $1 \leq a_1 < a_2 < \ldots < a_{50} < 100$. Show that $a_i + a_j = 99$ for some i and j.

   Proof. AG 33 $\square$

4. Show that in any group of 6 people you can find 3 who know each other or 3 who are strangers to each other.

Proof. Classical problem. See wiki on theorem on friends and strangers. ☐

5. Show that every convex polyhedron has two sides with the same number of edges. Can you give an example of a convex polyhedron with no 3 faces with the same number of edges?

Proof. AG 46 ☐

6. Inside a circle of radius 4 are 45 points. Show that you can find two of these points at most $\sqrt{2}$ apart. [Hint: Draw circles around each point.]

Proof. Variant of AG 44, but using circles of radius $\sqrt{2}/2$ around each point instead of squares. ☐

7. Let α be irrational and $\varepsilon > 0$.

   (a) Show that there exist two integers m and n such that $0 < |m\alpha - n| < \varepsilon$. Can you show that m can be taken to be positive with $m \leq \varepsilon^{-1}$? Can you deduce that the set of fractional parts $\{n\alpha\} = n\alpha - \lfloor n\alpha \rfloor$ is dense in $[0, 1)$, in the sense that they get arbitrarily close to any real in $[0, 1)$?

   (b) Deduce that for every sequence of digits $\overline{a_1 \ldots a_k}$ in base 10 some power of 2, when written in base 10, starts with this sequence of digits. E.g., $2^{1545} = 1234\ldots$.

   (c) (*) Can you find infinitely many n such that 2^n starts with the digits of n in base 10? E.g., among the numbers less than a million this is true for $6, 10, 1542, 77075, 113939$.

Proof. Classical Diophantine approximation problem. See wiki for Dirichlet approximation. ☐