Math 43900 Questionnaire

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I'd like to get to know the level of every student in class, your experience will problem solving and what kind of math you know. That way I can make sure you all know the basics and the most useful tricks and strategies.

Experience

1. Have you done problem solving before? If yes, how much? E.g., I'm interested if you've taken a class or seminar before, and when you first saw a math competition style problem.
2. Have you taken any math competitions before? Which?
3. Will you be able to take the Virginia Tech competition on October the 22nd, or will you take it at some other time?

Topics

For each of the following topics, please rate your knowledge. Answer options are: ??? meaning "never heard of it", 0 meaning "heard of it but don't know much", $\sim$ meaning "I have some exposure to the theory but not too much experience solving problems", ✓ meaning "I've seen a reasonably large amount of this". But really anything goes.

1. Complex numbers.
2. Polynomials.
3. Matrices and determinants.
4. Matrix eigenvalues.
5. Linear algebra.
6. Convergence of sequences and series.
7. Recurrently defined sequences.
8. Computing infinite sums and products.
9. Mathematical induction.
10. Pigeonhole principle.
11. Proof by contradiction.
12. Combinatorics.
13. Counting discrete math objects.

14. Invariants.
15. Inequalities.
16. AM-GM inequality.
17. Cauchy-Schwarz.
18. Generating functions.
19. Binomial coefficients and factorials.
20. The binomial expansion.
21. Functional equations.
22. Integrals.
23. Probability and expected values.
24. Games.
25. Geometry.
26. Number theory.
27. Modular arithmetic.
28. Graph theory.
29. Euler's "totient" function.
30. Chinese remainder theorem.
31. Fermat's little theorem.
32. Descent.
33. Taylor series.
34. Integer polynomials.
35. Writing integers in various digit bases.
36. Inclusion-exclusion principle.
37. Ordered sets and using maxima and minima.
38. Binary operations on sets (think composition laws in groups).
39. Limits.
40. Divisibility of polynomials.
41. Euclidean algorithm.
42. Permutations and their signs.
43. Jensen's inequality for convex functions.
44. Pythagorean triples.
45. Pell's equation.

46. Continued fractions.

47. Euler's formula for polyhedra.

48. Bayes' theorem for probabilities.

Math facts

Rate each of the following math facts from 1 to 5, 1 being "I've never seen this and I have no idea where it comes from" to 5 being "I've seen it and know how to do it".

1. $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$.

2. $X^m + 1 \mid X^n + 1$ if $m \mid n$.

3. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

4. The three heights in a triangle intersect.

5. The multiplicative group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ is cyclic if p is odd.

6. If X is a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and P is a polynomial then $P(X)$ has eigenvalues $P(\lambda_1), \ldots, P(\lambda_n)$.

7. If X and Y are matrices then XY and YX have the same trace.

8. If (x_n) is a sequence satisfying a recurrence relation of the form $x_n = a_1 x_{n-1} + \cdots a_d x_{n-d}$ for all $n \geq d$ then x_n has a general expression of the form $x_n = u_1\alpha_1^n + \cdots u_d\alpha_d^n$ where $\alpha_1, \ldots, \alpha_d$ are the roots of the polynomial $X^d - a_1X^{d-1} - \cdots - a_d = 0$.

9. If $P(X) \in \mathbb{Z}[X]$ is a *monic* polynomial with integer coefficients then every rational root of $P(X)$ must be an integer.

10. $\sum \frac{1}{p}$ diverges where p runs over the primes.

11. The Vandermonde identity

$$\det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{pmatrix} = \prod_{i>j} (x_i - x_j)$$

Math problems

For each of the following problems, spend a couple of minutes (by which I really mean 2 minutes on thinking about the problem, and not worrying about writing it up) to try to come up with a solution. Then rate your experience 1 to 5 with 1 = no idea where to start, 3 = have an idea, but the two minutes are up and I don't really know if it will work and 5 = know how to do it. Obviously use 2 and 4 for in between.

It's fine if all your answers are 1. It only means you haven't had enough exposure to these problems, and by the end of the semester you'll be able to do such problems very fast.

1. $\sqrt{2}$ and $\sqrt{2} + \sqrt{3}$ is irrational.

2. $a + b \geq 2\sqrt{ab}$.

3. $p \mid \binom{p}{k}$ if $1 \leq k \leq p-1$.

4. If A is a 2×2 matrix such that $A^2 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ then $a = 0$.

5. If you color every point in the plane in two colors there exists an equilateral triangle all of whose vertices have the same color.

6. How many integers less than 1000 are not divisible by 2 or 5?

7. Find all integers x, y such that $xy + 2x + 3y = 15$.

8. Show that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.