Math 20550 Calculus III Tutorial **Name:** ______________________________
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Tutorial Worksheet

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1. A particle moves with position function $\mathbf{r}(t) = \langle \cos t, \sin t, \cos^2 t\rangle$. Find the tangential and normal components of acceleration when $t = \pi/4$.

Solution: $\mathbf{r}'(t) = \langle -\sin t, \cos t, -2\cos t \sin t\rangle = \langle -\sin t, \cos t, -\sin(2t)\rangle$.

$\mathbf{r}''(t) = \langle -\cos t, -\sin t, -2\cos(2t)\rangle$.

At $t = \pi/4$, $\mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right\rangle$ and $\mathbf{r}''\left(\frac{\pi}{4}\right) = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\rangle$.

$$a_T = \frac{\mathbf{r}'\left(\frac{\pi}{4}\right) \cdot \mathbf{r}''\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|} = 0$$

Since $a_T = 0$, $\mathbf{a} = a_N \mathbf{N}$ and since $a_N \geqslant 0$, $a_N = \left|\mathbf{r}''\left(\frac{\pi}{4}\right)\right| = \left|\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\rangle\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1.$

2. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$.

Solution: The projection of the curve C onto the xy-plane is the curve $x^2 = 2y$ or $y = \frac{1}{2}x^2$, $z = 0$. Then we can choose the parameter $x = t$, and $y = \frac{1}{2}t^2$. Since C also lies on the surface $3z = xy$, we have $z = \frac{1}{3}xy = \frac{1}{3}(t)(\frac{1}{2}t^2) = \frac{1}{6}t^3$. Then parametric equations for C are $x = t$, $y = \frac{1}{2}t^2$, $z = \frac{1}{6}t^3$ and the corresponding vector equation is $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\right\rangle$. The origin corresponds to $t = 0$ and the point $(6, 18, 36)$ corresponds to $t = 6$, so

$$\begin{aligned} L &= \int_0^6 |\,\mathbf{r}'(t)\,|\,dt = \int_0^6 \left|\,\left\langle 1, t, \tfrac{1}{2}t^2\right\rangle\,\right| dt = \int_0^6 \sqrt{1^2 + t^2 + (\tfrac{1}{2}t^2)^2}\,dt \\ &= \int_0^6 \sqrt{1 + t^2 + \tfrac{1}{4}t^4}\,dt = \int_0^6 \sqrt{(1 + \tfrac{1}{2}t^2)^2}\,dt \\ &= \int_0^6 (1 + \tfrac{1}{2}t^2)\,dt = 42. \end{aligned}$$

3. Find the equation for the normal and osculating planes to the curves $\mathbf{r}(t) = (t - \frac{3}{2}\sin(t))\mathbf{i} + (1 - \frac{3}{2}\cos(t))\mathbf{j} + t\mathbf{k}$ at the point $(\pi, \frac{5}{2}, \pi)$.

Solution: First find t so that $\mathbf{r}(t) = \left\langle \pi, \frac{5}{2}, \pi \right\rangle$ so $\left\langle t - \frac{3}{2}\sin(t), 1 - \frac{3}{2}\cos(t), t \right\rangle = \left\langle \pi, \frac{5}{2}, \pi \right\rangle$. Clearly $t = \pi$ is the only possible solution from looking at the z-coordinates and this is a solution since $\sin(\pi) = 0$ and $\cos(\pi) = -1$.

$\mathbf{r}'(t) = \left\langle 1 - \frac{3}{2}\cos(t), \frac{3}{2}\sin(t), 1 \right\rangle$ and $\mathbf{r}'(\pi) = \left\langle \frac{5}{2}, 0, 1 \right\rangle$.

The vector $\mathbf{r}'(\pi)$ is a normal vector for the normal plane and $\left\langle \pi, \frac{5}{2}, \pi \right\rangle$ is a point so an equation is

$$\left\langle \frac{5}{2}, 0, 1 \right\rangle \bullet \langle x, y, z \rangle = \left\langle \frac{5}{2}, 0, 1 \right\rangle \bullet \left\langle \pi, \frac{5}{2}, \pi \right\rangle = \frac{7}{2}\pi$$

or $5x + 2z = 7\pi$.

A normal vector to the osculating plane is $\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)$.

$\mathbf{r}''(t) = \left\langle \frac{3}{2}\sin(t), \frac{3}{2}\cos(t), 0 \right\rangle$ and $\mathbf{r}''(\pi) = \left\langle 0, -\frac{3}{2}, 0 \right\rangle$.

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{5}{2} & 0 & 1 \\ 0 & -\frac{3}{2} & 0 \end{vmatrix} = \left\langle \frac{3}{2}, 0, -\frac{15}{4} \right\rangle$$

Hence an equation for the osculating plane is

$$\left\langle \frac{3}{2}, 0, -\frac{15}{4} \right\rangle \bullet \langle x, y, z \rangle = \left\langle \frac{3}{2}, 0, -\frac{15}{4} \right\rangle \bullet \left\langle \pi, \frac{5}{2}, \pi \right\rangle = -\frac{9}{4}\pi$$

or $2x - 5z = -3\pi$.

4. Find the unit tangent, the unit normal, and the binormal vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ to the curve $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, 3t^2 \rangle$ at $t = \pi$.

Solution:

$\mathbf{r}'(t) = \langle 2\cos(2t), -2\sin(2t), 6t \rangle$ and $\mathbf{r}'(\pi) = \langle 2, 0, 6\pi \rangle$.

$\mathbf{r}''(t) = \langle -4\sin(2t), -4\cos(2t), 6 \rangle$ and $\mathbf{r}''(\pi) = \langle 0, -4, 6 \rangle$.

$\mathbf{T}(t)$ and $\mathbf{r}'(t)$ point in the same direction so

$$\mathbf{T}(\pi) = \frac{1}{|\langle 2, 0, 6\pi \rangle|} \langle 2, 0, 6\pi \rangle = \frac{1}{\sqrt{4 + 36\pi^2}} \langle 2, 0, 6\pi \rangle = \frac{1}{\sqrt{1 + 9\pi^2}} \langle 1, 0, 3\pi \rangle$$

The binormal and $\mathbf{r}'(t) \times \mathbf{r}''(t)$ point in the same direction and

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 6\pi \\ 0 & -4 & 6 \end{vmatrix} = \langle 24\pi, -(12), -8 \rangle = \langle 24\pi, -12, -8 \rangle$$

Hence

$$\mathbf{B}(\pi) = \frac{1}{|\langle 24\pi, -12, -8 \rangle|} \langle 24\pi, -12, -8 \rangle = \frac{1}{|\langle 6\pi, -3, -2 \rangle|} \langle 6\pi, -3, -2 \rangle =$$

$$\frac{1}{\sqrt{36\pi^2 + 9 + 4}} \langle 6\pi, -3, -2 \rangle = \frac{1}{\sqrt{13 + 36\pi^2}} \langle 6\pi, -3, -2 \rangle$$

Finally $\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$ so

$$\mathbf{N}(\pi) = \frac{1}{\sqrt{1 + 9\pi^2}} \frac{1}{\sqrt{13 + 36\pi^2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6\pi & -3 & -2 \\ 1 & 0 & 3\pi \end{vmatrix} = \frac{1}{\sqrt{1 + 9\pi^2}} \frac{1}{\sqrt{13 + 36\pi^2}} \left\langle -9\pi, -(18\pi^2 + 2), 3 \right\rangle =$$

$$\frac{1}{\sqrt{1 + 9\pi^2}} \frac{1}{\sqrt{13 + 36\pi^2}} \left\langle -9\pi, -18\pi^2 - 2, 3 \right\rangle$$

5. Find equations of the normal and osculating planes of the curve $\mathbf{r}(t) = \langle t^2, \ln t, t \ln t\rangle$ at the point $(1, 0, 0)$.

Solution: From $\mathbf{r}(t) = \langle t^2, \ln t, t \ln t\rangle$, we have $\mathbf{r}'(t) = \langle 2t, 1/t, 1 + \ln t\rangle$ and $\mathbf{r}''(t) = \langle 2, -1/t^2, 1/t\rangle$. The point $(1, 0, 0)$ corresponds to $t = 1$, and $\mathbf{r}'(1) = \langle 2, 1, 1\rangle$, $\mathbf{r}''(1) = \langle 2, -1, 1\rangle$, $\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 2, 0, -4\rangle$. Since $\mathbf{r}'(1)$ is a normal vector to the normal plane, an equation for the normal plane is

$2(x - 1) + y + z = 0$ or $2x + y + z = 2$.

Since $\mathbf{r}'(1) \times \mathbf{r}''(1)$ is normal to the osculating plane, an equation for the osculating plane is

$2(x - 1) - 4z = 0$ or $2x - 4z = 2$.

6. Find equations of the normal and osculating planes of the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ at the point $(1, 1, 1)$

Solution: First we parametrize the curve of intersection. We can choose $y = t$; then $x = y^2 = t^2$ and $z = x^2 = t^4$, and the curve is given by $\mathbf{r}(t) = \langle t^2, t, t^4\rangle$. $\mathbf{r}'(t) = \langle 2t, 1, 4t^3\rangle$ and the point $(1, 1, 1)$ corresponds to $t = 1$, so $\mathbf{r}'(1) = \langle 2, 1, 4\rangle$ is a normal vector for the normal plane. Thus an equation of the normal plane is

$2(x - 1) + 1(y - 1) + 4(z - 1) = 0$ or $2x + y + 4z = 7$.

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4t^2+1+16t^6}} \langle 2t, 1, 4t^3\rangle$ and $\mathbf{T}'(t) = -\frac{1}{2}(4t^2+1+16t^6)^{-3/2}(8t+96)^5 \langle 2t, 1, 4t^3\rangle + (4t^2 + 1 + 16t^6)^{-1/2} \langle 2, 0, 12t^2\rangle$. A normal vector for the osculating plane is $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1)$, but $\mathbf{r}'(1) = \langle 2, 1, 4\rangle$ is parallel to $\mathbf{T}(1)$ and $\mathbf{T}'(1) = -\frac{1}{2}(21)^{-3/2}(104) \langle 2, 1, 4\rangle + (21)^{-1/2} \langle 2, 0, 12\rangle = \frac{2}{21\sqrt{21}} \langle -31, -26, 22\rangle$ is parallel to $\mathbf{N}(1)$ as is $\langle -31, -26, 22\rangle$, so $\langle 2, 1, 4\rangle \times \langle -31, -26, 22\rangle = \langle 126, -168, -21\rangle$ is normal to the osculating plane. Thus an equation for the osculating plane is $126(x - 1) - 168(y - 1) - 21(z - 1) = 0$ or $6x - 8y - z = -3$.