Math 20550 Calculus III Tutorial **Name:** ____________________

February 11, 2016

Tutorial Worksheet

Show all your work.

1. Do the following limits exist? If it exists, compute the limit, if not, explain why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}, \quad \lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4+y^4}$$

Solution. Along x-axis, $y=0$, $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{x\to 0} \frac{x^2 0^2}{x^4+3\cdot 0^4} = 0.$

Along the path $x=y$, $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{x\to 0} \frac{x^4}{4x^4} = \frac{1}{4}.$

Hence $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$ does not exist.

For the second limit, note that if $x=0$ or $y=0$, then the function is also zero. So we may assume that we are taking the limit over nonzero x and y. Then we can rewrite the limit as follows

$$\lim_{x,y\to 0} \frac{x^4y^4}{x^4+y^4} = \lim_{x,y\to 0} \frac{1}{\frac{x^4+y^4}{x^4y^4}} = \lim_{x,y\to 0} \frac{1}{\frac{1}{x^4}+\frac{1}{y^4}} = \frac{1}{\infty} = 0$$

□

2. Let $f(x,y)$ be a function whose partial derivatives satisfy

$$\partial_x f(x,y) = x+4y$$

$$\partial_y f(x,y) = 3x-y$$

Find f.

Solution. There is no such function. Indeed, the first condition implies that

$$f(x,y) = \frac{1}{2}x^2 + 4xy + g(y)$$

for some function $g(y)$ in the variable y. Similarly, the second condition implies that

$$f(x,y) = 3xy - \frac{1}{2}y^2 + h(x)$$

for some function $h(x)$ in the variable x. Equating these two expressions yields

$$h(x) - g(y) = \frac{1}{2}(x+y)^2 = \frac{1}{2}(x^2+y^2+xy).$$

The left hand side has no mixed terms, while the right hand side has an xy, which is impossible.

Alternatively, we could have shown this by appealing to Clariut's theorem. Namely, we see from the conditions that

$$\partial_y \partial_x f = 4$$

while

$$\partial_x \partial_y f = 3$$

Clairut's theorem dictates then that $3 = 4$, which is false. Thus there cannot be a function with the above partial derivatives. □

3. Determine the subset of $\mathbb{R}^2$ on which the function

$$f(x, y) = \ln(x^2 + y^2 - 4)$$

is continuous and compute its partial derivatives.

Solution. Since the function $\ln(x)$ is smooth, the above $f(x, y)$ is continuous wherever the expression $x^2 + y^2 - 4$ takes values in the domain of $\ln(x)$, which is positive real numbers. Thus the function is continuous wherever it is defined, which is the subset

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 4\}$$

Also

$$\partial_x f(x, y) = \frac{2x}{x^2 + y^2 - 4}$$

and

$$\partial_y f(x, y) = \frac{2y}{x^2 + y^2 - 4}$$

□

4. A cylinder containing an incompressible fluid is being squeezed from both ends. If the length of the cylinder is changing at a rate of -3m/s, calculate the rate at which the radius is changing when the radius is 2m and the length is 1m. (Note: An incompressible fluid is a fluid whose volume does not change.)

Solution. The volume of the cylinder is $V = \pi r^2 l$. By the assumption of incompressibility $\frac{dV}{dt} = 0$.

$$\begin{aligned} 2\pi r l \frac{dr}{dt} + \pi r^2 \frac{dl}{dt} &= 0 \\ 2\pi r l \frac{dr}{dt} - 3\pi r^2 &= 0 \\ \frac{dr}{dt} &= \frac{3r}{2l}. \end{aligned}$$

When $r = 2, l = 1$, $\frac{dr}{dt} = 3$m/s.

□