Math 20550 Calculus III Tutorial **Name:** ______________________________
February 25, 2016

Tutorial Worksheet

Show all your work.

1. The height of a mountain is given by $f(x, y) = 8000 - \frac{x^2}{100} - \frac{y^2}{50}$. Suppose one is at the point $(60, 100)$. In what direction is the elevation decreasing fastest? What is the maximum rate of change of the elevation at this point?

Solution: We compute $\nabla f = \langle -0.02x, -0.04y \rangle$. The maximum rate of change of the elevation will then occur in the direction of $\nabla f(60, 100) = \langle -1.2, -4 \rangle$. The direction in which the elevation is decreasing fastest is $\langle 1.2, 4 \rangle$. The maximum rate of change of the elevation at this point is $\sqrt{(-1.2)^2 + (-4)^2} = \sqrt{17.44}$.

2. Find the tangent plane and the normal line to the surface $x^2 + y^2 + z^2 = 3x$ at the point $P : (1, 1, 1)$. Also find the tangent line to the curve of the intersection of this surface and $2x - 3y + 5z - 4 = 0$ at P.

Solution: The given surface is the zero level surface of the function $f(x, y, z) = x^2 + y^2 + z^2 - 3x$, whose gradient is

$$\nabla f(x, y, z) = \langle 2x - 3, 2y, 2z \rangle.$$

Thus, at the point $(1, 1, 1)$, we have $\nabla f(1, 1, 1) = \langle -1, 2, 2 \rangle$. This vector is a normal vector to the tangent plane and a direction vector for the normal line, so that an equation of the tangent plane at $(1, 1, 1)$ is

$$(-1)(x - 1) + 2(y - 1) + 2(z - 1) = 0 \;\Rightarrow\; -x + 2y + 2z = 3,$$

and an equation for the normal line at $(1, 1, 1)$ is

$$r(t) = \langle 1, 1, 1 \rangle + t\langle -1, 2, 2 \rangle = \langle 1 - t, 1 + 2t, 1 + 2t \rangle.$$

We compute $\langle -1, 2, 2 \rangle \times \langle 2, -3, 5 \rangle = \langle 16, 9, -1 \rangle$. The tangent line to the intersection curve at $(1, 1, 1)$ is $\langle 1 + 16t, 1 + 9t, 1 - t \rangle$.

3. Find the local maxima, minima, and saddle points of the function $z = (x^2 + y^2)e^{-y}$.

Solution: Compute the gradient of $z = z(x, y)$, and then set it equal to zero to get:

$$\nabla z(x, y) = \langle 2xe^{-y}, -x^2e^{-y} + 2ye^{-y} - y^2e^{-y} \rangle = \langle 0, 0 \rangle.$$

The only critical points are $(0, 0)$ and $(0, 2)$. At these points, we compute the Hessian of z:

$$\text{Hess}_z(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \text{Hess}_z(0, 2) = \begin{pmatrix} 2e^{-2} & 0 \\ 0 & -2e^{-2} \end{pmatrix}.$$

Note that at $(0, 2)$, the determinant of the Hessian is **negative**, which means it is a saddle point. At $(0, 0)$, the determinant is **positive** and the **first entry of the first row is positive**, which means it is a local minimum.

4. Identify the maximum and minimum values attained by $z = x^2y - 2x^2$ within the triangle T bounded by the points $P(0,0)$, $Q(2,0)$, and $R(0,4)$.

Solution: First, we check for critical points in the interior of the triangle. The gradient of z is $\nabla z(x,y) = \langle 2xy - 4x, x^2\rangle$, but this only vanishes along the boundary of the triangle (where $x = 0$), so we move on to analyze the boundary.
From P to Q, the function z equals $-2x^2$, $0 \le x \le 2$, which has a maximum of 0 at 0 and a minimum of -8 at 2.
From P to R, the function z is identically zero.
From Q to R, $y = -2x + 4$, so the function z becomes $z = x^2(-2x+4) - 2x^2 = -2x^3 + 2x^2$, $0 \le x \le 2$. Using Calc I tools, we discover that in this interval, z has a minimum of -8 at $x = 2$ and a maximum of $8/27$ at $x = 2/3$.
Comparing all these results, we conclude that on the whole triangle (including boundaries), the function reaches a global maximum of $8/27$ at $(2/3, 8/3)$ and a global min of -8 at $(2,0)$.
5. Identify the maximum and minimum values attained by $z = 4x^2 - y^2 + 1$ within the region R bounded by the curve $4x^2 + y^2 = 16$.

Solution: First, we check for critical points in the interior of the region. We have $\nabla z = \langle 8x, -2y\rangle$, so the only critical point is $(0,0)$; but the Hessian at $(0,0)$ has negative determinant, hence $(0,0)$ is not a maximum, nor a minimum.
Now, we check for critical points along the boundary. The boundary consists of those points (x,y) such that $g(x,y) = 0$, where $g(x,y) = 4x^2 + y^2 - 16$. So $z = 4x^2 - (16 - 4x^2) + 1 = 8x^2 - 15, x \in [-2,2]$. Then we see readily $z_{max} = 17$ and $z_{min} = -15$.
6. Use Lagrange multiplier to maximize $f(x,y,z) = xyz$ subject to $x^2 + 2y^2 + 3z^2 = 9$, assuming that x, y, and z are nonnegative. Explain why the extremum you find is a maximum.

Solution: The gradient of f is

$$\nabla f = \langle yz, xz, xy\rangle .$$

Let $g = x^2 + 2y^2 + 3z^2$, then $\nabla g = \langle 2x, 4y, 6z\rangle$. The system of equations we get by Lagrange multipliers is thus

$$yz = 2\lambda x, xz = 4\lambda y, xy = 6\lambda z, x^2 + 2y^2 + 3z^2 = 9$$

If we assume x, y, z are nonzero we then have

$$\lambda = \frac{yz}{2x} = \frac{xz}{4y} = \frac{xy}{6z}$$

$$x^2 + 2y^2 + 3z^2 = 3x^2 = 9 \implies x^2 = 3 \implies x = \pm\sqrt{3}$$

Then we have $y = \pm\sqrt{\frac{3}{2}}$ and $z = \pm 1$. So we get 8 solutions.

$(\sqrt{3}, \sqrt{\frac{3}{2}}, 1), (-\sqrt{3}, \sqrt{\frac{3}{2}}, 1), (\sqrt{3}, -\sqrt{\frac{3}{2}}, 1), (\sqrt{3}, \sqrt{\frac{3}{2}}, -1), (\sqrt{3}, -\sqrt{\frac{3}{2}}, -1),$
$(-\sqrt{3}, -\sqrt{\frac{3}{2}}, 1), (-\sqrt{3}, \sqrt{\frac{3}{2}}, -1), (-\sqrt{3}, -\sqrt{\frac{3}{2}}, -1).$

If $x = 0$, then we get solutions $(0,0,\sqrt{3}), (0,0,-\sqrt{3}), (0, \sqrt{\frac{9}{2}}, 0), (0, -\sqrt{\frac{9}{2}}, 0)$ (in those cases $\lambda = 0$.) If $y = 0$, then we get solutions $(0,0,\sqrt{3}), (0,0,-\sqrt{3}), (3,0,0), (-3,0,0)$ (in those cases $\lambda = 0$.) If $z = 0$, then we get solutions $(0, \sqrt{\frac{9}{2}}, 0), (0, -\sqrt{\frac{9}{2}}, 0), (3,0,0), (-3,0,0)$ (in those cases $\lambda = 0$.) Evaluate $f(x,y,z) = xyz$ at those points and we have 3 possible outputs $\pm\frac{3}{\sqrt{2}}, 0$. Hence we conclude that the maximum is $\frac{3}{\sqrt{2}}$ and in particular it is attained at $(\sqrt{3}, \sqrt{\frac{3}{2}}, 1)$