Math 20550 Calculus III Tutorial **Name:** ______________________________
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Tutorial Worksheet

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1. Find the maximum value of the function $f(x, y, z) = x + 2y$ on the curve of intersection of the plane $x + y + z = 1$ and the cylinder $y^2 + z^2 = 4$.

Solution: Basically, the problem asks to maximize f subject to two constraints:

$$g(x, y, z) = x + y + z = 1$$
$$h(x, y, z) = y^2 + z^2 = 4$$

Firstly, compute

$$\nabla f(x, y, z) = \langle 1, 2, 0\rangle$$
$$\nabla g(x, y, z) = \langle 1, 1, 1\rangle$$
$$\nabla h(x, y, z) = \langle 0, 2y, 2z\rangle$$

By the method of Lagrange Multipliers, for some scalars λ, μ, we have

$$1 = \lambda$$
$$2 = \lambda + 2\mu y$$
$$0 = \lambda + 2\mu z$$
$$x + y + z = 1$$
$$y^2 + z^2 = 4$$

By solving the equations, we obtain the points $(1, -\sqrt{2}, \sqrt{2})$ and $(1, \sqrt{2}, -\sqrt{2})$.
So then,

$$f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$$
$$f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}$$

Thus, the maximum value of f is $1 + 2\sqrt{2}$ on the curve of intersection.

2. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Solution: We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$.

$\nabla f = \langle 2x, 2y, 2z\rangle$, $\nabla g = \langle 1, 1, 2\rangle$ and $\nabla h = \langle 2x, 2y, -1\rangle$.

Thus, we need

$$2x = \lambda + 2\mu x$$
$$2y = \lambda + 2\mu y$$
$$2z = 2\lambda - \mu$$
$$x + y + 2z = 2$$
$$x^2 + y^2 - z = 0.$$

By solving the equations, we obtain two points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(-1, -1, 2)$. Then we have $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the point on $(-1, -1, 2)$ is the one farthest from the origin.

3. (a)Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1, 2] \times [0, 3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower left corners.

(b)Use the Midpoint Rule to estimate the volume in part(a).

Solution: (a)The surface is the graph of $f(x, y) = 1 + x^2 + 3y$ and $\Delta A = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$, so we estimate

$$V = \iint_R (1 + x^2 + 3y) dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*) \Delta A$$

$$= f(1, 0)\Delta A + f(1, \frac{3}{2})\Delta A + f(\frac{3}{2}, 0)\Delta A + f(\frac{3}{2}, \frac{3}{2})\Delta A$$

$$= 2(\frac{3}{4}) + \frac{13}{2}(\frac{3}{4}) + \frac{13}{4}(\frac{3}{4}) + \frac{31}{4}(\frac{3}{4}) = \frac{117}{8} = 14.625$$

(b)

$$V = \iint_R (1 + x^2 + 3y) dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{x}_j) \Delta A$$

$$= f(\frac{5}{4}, \frac{3}{4})\Delta A + f(\frac{5}{4}, \frac{9}{4})\Delta A + f(\frac{7}{4}, \frac{3}{4})\Delta A + f(\frac{7}{4}, \frac{9}{4})\Delta A$$

$$= \frac{77}{16}(\frac{3}{4}) + \frac{149}{16}(\frac{3}{4}) + \frac{101}{16}(\frac{3}{4}) + \frac{173}{16}(\frac{3}{4}) = \frac{375}{16} = 23.4375$$

4. Evaluate the double integral $\iint_R (4 - 2y) dA$, $R = [0, 1] \times [0, 1]$ by identifying it as the volume of a solid.

Solution: $z = f(x, y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0, 1] \times [0, 1] \times [0, 4]$ which lies below the plane $z = 4 - 2y$.

So

$$\iint_R (4 - 2y) dA = (1)(1)(2) + \frac{1}{2}(1)(1)(2) = 3$$

5. Calculate the iterated integral

(a) $\int_0^2 \int_0^\pi r\sin^2(\theta)d\theta dr$

(b) $\int\int_R ye^{-xy}dA,\ R = [0,2]\times[0,3]$

Solution:

(a) $\int_0^2 \int_0^\pi r\sin^2\theta d\theta dr = \int_0^2 rdr \int_0^\pi \sin^2\theta d\theta = \int_0^2 rdr \int_0^\pi \frac{1}{2}(1-\cos 2\theta)\theta d\theta$

$= [\frac{1}{2}r^2]_0^2 \cdot \frac{1}{2}[\theta - \frac{1}{2}\sin 2\theta]_0^\pi = \pi$

(b) $\int\int_R ye^{-xy}dA = \int_0^3 \int_0^2 ye^{-xy}dxdy = \int_0^3 [-e^{-xy}]_{x=0}^{x=2}dy = \int_0^3(-e^{-2y}+1)dy = [\frac{1}{2}e^{-2y}+y]_0^3$

$= \frac{1}{2}e^{-6} + \frac{5}{2}$

6. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

Solution: The cylinder intersects the xy-plane along the line $x = 4$, so in the first octant, the solid lies below the surface $z = 16 - x^2$ and above the rectangle $R = [0,4]\times[0,5]$ in the xy-plane.

$V = \int_0^5 \int_0^4 (16-x^2)dxdy = \int_0^4 (16-x^2)dx \int_0^5 dy = [16x - \frac{1}{3}x^3]_0^4[y]_0^5 = \frac{640}{3}$