Worksheet 7 Solutions, Math 10560

1. Show that
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dA = \pi$$
and deduce that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

<u>*Solution*</u>:

We convert to polar coordinates, remembering that $dx\,dy$ becomes $r\,dr\,d\theta$:

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

The extra r allows us to use u-substitution. We take $u = r^2$, so that $du = 2r\,dr$. This yields

$$\pi \int_0^{\infty} e^{-u}\,du = \pi\, e^{-u}\Big|_0^{\infty} = \pi$$

In rectangular coordinates, we can rewrite the integral as

$$\int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy$$

whereby

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

2. Find the mass of the tetrahedron with points given by $(0,0,0), (2,0,0), (0,3,0), (0,0,1)$ with density function $\rho(x,y,z) = x$.

   Solution:

   First, we need to find the equation of the plane containing the points $(2,0,0), (0,3,0)$ and $(0,0,1)$. The normal vector is given by the cross product of the two vectors

   $$\langle 2, 0, -1\rangle \times \langle 0, 3, -1\rangle = \langle 3, 2, 6\rangle$$

   which gives the equation for the plane as

   $$3x + 2y + 6z = 6$$

   Thus, the triple integral computing the mass is given by

   $$m = \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{1-\frac{1}{2}x-\frac{1}{3}y} x\, dz\, dy\, dx$$

   Immediately, this becomes

   $$\int_0^2 \int_0^{3-\frac{3}{2}x} x\left(1 - \frac{1}{2}x - \frac{1}{3}y\right) dy\, dx$$

   which becomes

   $$\int_0^2 \frac{3}{8}x(x-2)^2\, dx = \frac{1}{2}$$

3. Find the mass and center of mass of the solid E given by $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$ and constant density ρ.

   <u>*Solution*:</u>

   We will just set up the integrals. Once we set up the integral it will be easy to figure out the integrals for the moments. Supposing that z and y are fixed, then x-coordinate can vary between the yz-plane and the pane given by $x + z = 1$, i.e. x can vary between 0 and $1 - z$. This tells us that the first part of the triple integral for the mass will be given by

$$\int_0^{1-z} \rho \, dx$$

   Next, suppose that y has been fixed. Then z can vary between 0 and the value $1 - y^2$. This tells us that the next part of the triple integral is

$$\int_0^{1-y^2} \int_0^{1-z} \rho \, dx \, dz$$

   Since y just varies between -1 and 1, the triple integral for the mass is

$$m = \int_{-1}^{1} \int_0^{1-y^2} \int_0^{1-z} \rho \, dx \, dz \, dy$$

   The formulas for the moments are then given by the usual formulas, i.e. the x-moment is

$$M_{yz} = \int_{-1}^{1} \int_0^{1-y^2} \int_0^{1-z} \rho x \, dx \, dz \, dy$$

4. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ for $a > 0$, assuming that S has constant density K.

   <u>*Solution*:</u>

   We will use cylindrical coordinates. By the symmmetry of the paraboloid, θ always varies between 0 and 2π. For a fixed z, the radius r can vary between 0 and $\frac{1}{2}\sqrt{z}$, since the upper bound for r is given by $z = 4r^2$. So the integral for the mass is

$$\int_0^{2\pi} \int_0^{a} \int_0^{\frac{1}{2}\sqrt{z}} Kr \, dr \, dz \, d\theta = 2\pi \int_0^{a} \frac{1}{2} K \, r^2 \Big|_0^{\frac{1}{2}\sqrt{z}} dz = \frac{\pi}{4} \int_0^{2} Kz \, dz = \frac{Ka^2\pi}{8}$$

   The symmetry of the volume tells us that the center of mass will have x- and y-components zero. So we just need to compute

$$M_{xy} = \int_0^{2\pi} \int_0^{a} \int_0^{\frac{1}{2}\sqrt{z}} Kzr \, dr \, dz \, d\theta = \frac{K\pi a^3}{12}$$

   So that the center of mass is

$$(0, 0, \frac{2a}{3})$$