Math 20550 Calculus III Tutorial **Name:** ______________________________
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Tutorial Worksheet

Show all your work.
1. Evaluate (using spherical coordinates)

$$\iiint_E dV$$

where E is the solid that lies within $(x^2+y^2+z^2)^2 = 8z$.

Solution: Write the equation $(x^2+y^2+z^2)^2 = 8z$ in spherical coordinate one gets

$$\rho = \sqrt[3]{8\cos\phi}.$$

On and within the surface $(x^2+y^2+z^2)^2 = 8z$, we have $z \geq 0$. And also note the surface passes through the origin. Therefore one concludes that the limit for ρ is $0 \leq \rho \leq 2\sqrt[3]{\cos\phi}$ and the limit for ϕ is $0 \leq \phi \leq \frac{\pi}{2}$. Therefore we compute

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\sqrt[3]{\cos\phi}} \rho^2 \sin\phi d\rho d\phi d\theta = \frac{8\pi}{3}.$$

2. Compute the volume of the solid defined by

$$x^2+y^2+z^2-2z \leq 0$$

and

$$x^2+y^2 \leq \frac{3}{2}z.$$

(Use triple integrals in spherical coordinates. You can use the fact $\int \frac{\cos^3 x}{\sin^5 x}dx = -\frac{\cot^4 x}{4} + C$.)

Solution: Rewrite the inequalities in spherical coordinate we see the solid is

$$\rho \geq 0, \rho \leq 2\cos\phi, \rho \leq \frac{3\cos\phi}{2\sin^2\phi}.$$

When $0 \leq \phi \leq \frac{\pi}{3}$ we have $\frac{3\cos\phi}{2\sin^2\phi} \geq 2\cos\phi$ and when $\frac{\pi}{3} \leq \phi \leq \frac{\pi}{2}$ we have $2\cos\phi \geq \frac{3\cos\phi}{2\sin^2\phi}$. Therefore we have

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta$$

$$+ \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{\frac{3\cos\phi}{2\sin^2\phi}} \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \frac{5\pi}{4} + \frac{\pi}{16} = \frac{21\pi}{16}.$$

3. Let the parallelogram D be defined by

$$5 \geq x + 2y \geq 2,$$

$$1 \geq y - x \geq -2.$$

Compute

$$\iint_D 2dA.$$

(Hint: Use change of variable: $u = x + 2y, v = y - x$. So $x = \frac{u-2v}{3}, y = \frac{u+v}{3}$.)

Solution: Let $u = x + 2y, v = y - x$. So $x = \frac{u-2v}{3}, y = \frac{u+v}{3}$.

$$|\frac{\partial(x, y)}{\partial(u, v)}| = \frac{1}{3}.$$

So by change of variable we get

$$\int_2^5 \int_{-2}^1 \frac{2}{3} dv\, du = 6.$$

4. Let D be the region in the first quadrant that is defined by

$$1 \geq y^2 - x^2 \geq 0,$$

$$4 \geq xy \geq 3.$$

Use change of variable to compute the double integral

$$\iint_D (y^2 - x^2)^{xy}(x^2 + y^2)dA.$$

(Hint: let $u = y^2 - x^2, v = xy$. Using implicit differentiation we can obtain (try verifying one of them) $\frac{\partial x}{\partial u} = -\frac{x}{2(y^2+x^2)}, \frac{\partial y}{\partial u} = \frac{y}{2(y^2+x^2)}, \frac{\partial x}{\partial v} = \frac{y}{x^2+y^2}, \frac{\partial y}{\partial v} = \frac{x}{x^2+y^2}$.)

Solution: Let $u = y^2 - x^2, v = xy$.

Using implicit differentiation we obtain $\frac{\partial x}{\partial u} = -\frac{x}{2(y^2+x^2)}, \frac{\partial y}{\partial u} = \frac{y}{2(y^2+x^2)}$, $\frac{\partial x}{\partial v} = \frac{y}{x^2+y^2}, \frac{\partial y}{\partial v} = \frac{x}{x^2+y^2}$.

$$|\frac{\partial(x, y)}{\partial(u, v)}| = \frac{1}{2(x^2 + y^2)}.$$

So by change of variable, the integral is

$$\frac{1}{2} \int_3^4 \int_0^1 u^v du dv = \frac{1}{2} \ln \frac{5}{4}.$$