Math 20550 Calculus III Tutorial **Name:** ______________________________
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Tutorial Worksheet

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1. Evaluate $\int_C x^2dx + y^2dy$, where C consists of the arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$ followed by the line segment frome $(0, 2)$ to $(4, 3)$.

Solution: We need to parametrize the curve. First, setting $C = C_1 + C_2$, where C_1 is the arc of the circle, and C_2 is the line segment.

On C_1: $x = 2\cos t$, $dx = -2\sin tdt$, $y = 2\sin t$, $dy = 2\cos tdt$, $0 \leq t \leq \frac{\pi}{2}$.

On C_2: $x = 4t$, $dx = 4dt$, $y = 2 + t$, $dy = dt$, $0 \leq t \leq 1$.

Then

$$\int_C x^2dx + y^2dy = \int_{C_1} x^2dx + y^2dy + \int_{C_2} x^2dx + y^2dy$$

$$= \int_0^{\frac{\pi}{2}} (2\cos t)^2(-2\sin tdt) + (2\sin t)^2(2\cos tdt) + \int_0^1 (4t)^2(4dt) + (2+t)^2dt$$

$$= 8\int_0^{\frac{\pi}{2}} (-\cos^2 t\sin t + \sin^2 t\cos t)dt + \int_0^1 (65t^2 + 4t + 4)dt = \frac{83}{3}$$

2. Compute $\int_C x^2ds$, C is the intersection of the surface $x^2 + y^2 + z^2 = 4$ and the plane $z = \sqrt{3}$.

Solution: We need to parametrize the curve. First, setting $z = \sqrt{3}$ in the equation of the surface, we get $x^2 + y^2 + 3 = 4$ or $x^2 + y^2 = 1$, which is a circle of radius 1. Therefore, we can parametrize x and y as a circle, with z as a constant $\sqrt{3}$.

Let $x = \cos t$, $y = \sin t$, $z = \sqrt{3}$, $0 \leqslant t \leqslant 2\pi$.

$$ds = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt = dt$$

$$\int_C x^2ds$$

$$= \int_0^{2\pi} \cos^2 tdt$$

$$= 4\int_0^{\pi/2} \cos^2 tdt$$

$$= \pi$$

3. Evaluate $\int_C \nabla f d\mathbf{r}$ where $f(x,y,z) = \cos \pi x + \sin \pi y - xyz$ and C is any path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2,1,-1)$.

Solution: By the Fundamental Theorem for the line integral

$$\int_C \nabla f d\mathbf{r}$$

$$= f(2,1,-1) - f\left(1, \frac{1}{2}, 2\right)$$

$$= \cos 2\pi + \sin \pi + 2 - \left(\cos \pi + \sin \frac{\pi}{2} - 1\right)$$

$$= 4$$

4. Find a function f such that $\mathbf{F} = \nabla f$ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C. Where $\mathbf{F}(x,y,z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$, C: $x = \sqrt{t}$, $y = t+1$, $z = t^2$, $0 \le t \le 1$.

Solution: $f_x(x,y,z) = y^2z + 2xz^2$ implies $f(x,y,z) = xy^2z + x^2z^2 + g(y,z)$ and so $f_y(x,y,z) = 2xyz + g_y(y,z)$. But $f_y(x,y,z) = 2xyz$ so $g_y(y,z) = 0$. Just let $g(y,z) = h(z)$. Thus, $f(x,y,z) = xy^2z + x^2z^2 + h(z)$ and $f_z(x,y,z) = xy^2 + 2x^2z + h'(z)$. But $f_z(x,y,z) = xy^2 + 2x^2z$, so $h'(z) = 0$. $h(z) = K$. Hence $f(x,y,z) = xy^2z + x^2z^2$(taking $K = 0$).

$t = 0$ corresponds to the point $(0,1,0)$ and $t = 1$ corresponds to $(1,2,1)$, so by the Fundamental Theorem for line integrals, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2,1) - f(0,1,0) = 5$$