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Worksheet 7, Math 10560

1. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle x^2+y, 3x-y^2\rangle$ and C is the positively oriented boundary curve of a region D whose area is 8.

   Solution: We will use Green's theorem to compute this integral. Note that $P(x,y) = x^2+y$ and $Q(x,y) = 3x-y^2$. So by Green's theorem we get

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \partial_x Q - \partial_y P\, dA = \iint_D 3-1\, dA = 2\cdot 8 = 16$$

   where we use the fact that $\iint_D dA = 8$.

2. Consider the vector field $\mathbf{F} = \langle y, -x, 0\rangle$. Draw a picture of this vector field and determine if this vector field is conservative. Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the unit circle.

   Solution: The vector field is not conservative as its curl is not zero:

   $$\nabla\times\mathbf{F} = \langle 0, 0, \partial_x(-x) - \partial_y(y)\rangle = \langle 0,0,-2\rangle$$

   To compute the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ we will use Green's theorem. Note that the the z-component of the curl $\nabla\times\mathbf{F}$ is used in Green's theorem

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D -2\, dA = -2\pi$$

   where D is the unit disc.

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3. Compute the line integral $\int_C xy^2dx + 2x^2y\,dy$ when C is the triangle with vertices $(0,0),(2,2)$ and $(2,4)$.

   Solution: Note that $P(x,y) = xy^2$ and $Q(x,y) = 2x^2y$. So to use Green's theorem we compute
   $$\partial_x Q - \partial_y P = 4xy - 2xy = 2xy$$
   so that
   $$\int_C xy^2dx + 2x^2y\,dy = \iint_T 2xy\,dA$$
   where T denotes the interior of C. Expanding the right hand side gives
   $$\iint_T 2xy\,dA = \int_0^2 \int_x^{2x} 2xy\,dy\,dx = \int_0^2 xy^2\Big|_x^{2x}\,dx = \int_0^2 3x^3\,dx = \frac{3}{4}8 = 6$$

4. Show that for a function f and a vector field $\mathbf{F}$ we have the following 'product rule' for the divergence
   $$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$$

   Solution: Let us write $\mathbf{F} = \langle P, Q, R\rangle$. We start by expanding $\nabla \cdot (f\mathbf{F})$. Note that $f\mathbf{F} = \langle fP, fQ, fR\rangle$. So
   $$\nabla \cdot (f\mathbf{F}) = \partial_x(fP) + \partial_y(fQ) + \partial_x(fR)$$
   Using the Leibniz rule gives
   $$\nabla \cdot (f\mathbf{F}) = (\partial_x f)P + f\partial_x P + (\partial_y f)Q + f\partial_y Q + (\partial_z f)R + f\partial_z R$$
   Rearranging these terms gives
   $$\nabla \cdot (f\mathbf{F}) = [\partial_x(f)P + (\partial_y f)Q + (\partial_z f)R] + f\,[\partial_x P + \partial_y Q + \partial_z R]$$
   We easily identify the first term as $\nabla(f) \cdot \mathbf{F}$ and the second term as $f\nabla \cdot \mathbf{F}$. This gives the desired equality.