Math 20550 Calculus III Tutorial **Name:** ______________________________
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Tutorial Worksheet

Show all your work.

1. Let S be the surface $y^2+z^2=1$, $z \geq 0$ and $0 \leq x \leq 1$. Let S have the upward orientation. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = \langle 0,0,z \rangle$.

Solution: We use x, y as parameters. The surface then is given by

$$r(x,y) = \langle x, y, \sqrt{1-y^2} \rangle, 0 \leq x \leq 1, -1 \leq y \leq 1.$$

$r_x = \langle 1,0,0 \rangle, r_y = \langle 0,1,-y(1-y^2)^{-\frac{1}{2}} \rangle$. Then we compute $r_x \times r_y = \langle 0, y(1-y^2)^{-\frac{1}{2}}, 1 \rangle$. So the flux integral is

$$\int_0^1 \int_{-1}^1 \langle 0,0,\sqrt{1-y^2} \rangle \cdot \langle 0, y(1-y^2)^{-\frac{1}{2}}, 1 \rangle dydx$$

$$= \int_0^1 \int_{-1}^1 \sqrt{1-y^2} dydx = \frac{\pi}{2}.$$

(That $\int_{-1}^1 \sqrt{1-y^2} dy = \pi/2$ can be seen immediately as it is the area of a half unit circle.)

2. Compute the surface area of the surface $x^2+y^2+z=4$ above xy-plane.

Solution: We parametrize the surface using polar coordinate and surface is thus given by

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 4-r^2 \rangle$$

$\mathbf{r}_r = \langle \cos\theta, \sin\theta, -2r \rangle$, $\mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$. $|\mathbf{r}_r \times \mathbf{r}_\theta| = |\langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle| = r\sqrt{4r^2+1}$. So the area is

$$\int_0^{2\pi} \int_0^2 r\sqrt{4r^2+1} drd\theta = \frac{17\sqrt{17}-1}{6}\pi.$$

3. Let the surface be given by $r(u,v) = \langle u\cos v, u\sin v, v\rangle$. Find the tangent plane of this surface at $p = (\sqrt{2}, \sqrt{2}, \pi/4)$.

Solution: At the point $u = 2, v = \pi/4$.
$r_u|_p = \langle \cos v, \sin v, 0\rangle|_p = \langle \sqrt{2}/2, \sqrt{2}/2, 0\rangle$.
$r_v|_p = \langle -u\sin v, u\cos v, 1\rangle|_p = \langle -\sqrt{2}, \sqrt{2}, 1\rangle$.
Then we compute $\langle \sqrt{2}/2, \sqrt{2}/2, 0\rangle \times \langle -\sqrt{2}, \sqrt{2}, 1\rangle = \langle \sqrt{2}/2, -\sqrt{2}/2, 2\rangle$.
Hence an equation for the tangent plane is

$$(x - \sqrt{2}) - (y - \sqrt{2}) + 2\sqrt{2}(z - \pi/4) = 0.$$

4. Evaluate $\iint_S \frac{1}{z} dS$ where S is given by $x^2 + y^2 + z^2 = 4, z \geq 1$.

Solution: We parametrize the surface using spherical coordinate and thus obtain

$$r(\phi, \theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi), 0 \leq \phi \leq \pi/3.$$

Thus $|r_\theta \times r_\phi| = 4\sin\phi$. So the integral is

$$\int_0^{2\pi} \int_0^{\pi/3} 2\tan\phi d\phi d\theta = 4\pi \ln 2.$$