Math 43900 Problem Solving
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Lecture 10 Sequences and series

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These problems are taken from the textbook, from Engels' *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Sequences and series

Overview

Calculus is certainly a vast topic in problem solving and you'll have to use pretty much everything you know from calculus to solve typical problems. Alas there's no way around algebraic manipulations and often an easy problem seems impossibly unless you see some weird algebraic manipulation. Nevertheless there are some tricks and some useful facts that can help you approach problems.

Typically problems with sequences involve either

1. finding the general term of a sequence of proving something about the general term of a sequence or
2. computing the limit of a sequence

A related topic is that of infinite series and infinite products where again I can identify two things:

3. showing that some series or products converge and
4. computing the values of some converging series or product.

Typically when asked to compute the value of a series or a product the actual computation might be easier than showing that the series converges and what you are doing is actually sensible. This is often true when having to compute integrals (a later topic) and you use Taylor expansions to compute the value, but showing convergence is harder.

Basic results

Remember all the convergence tests from calculus: bounded monotonous sequence test, ratio test and harmonic series test for series, telescoping sums and products, etc. In addition, the following are useful:

1. Cauchy's criterion: a sequence (x_n) converges if and only if as m and n grow, $|x_m - x_n|$ becomes small.
2. Stirling's approximation: $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\frac{\theta_n}{12n}}$ for some $\theta_n \in (0,1)$. (See textbook section 3.2.11.)
3. Recall that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.
4. You can go between infinite products and infinite series using logarithms and exponentiations.

2 Problems

2.1 Sequences and their limits

Easier

1. Compute $\lim \sqrt[n]{n}$.
2. Compute the limit $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$.
3. Suppose $a \in \mathbb{Z}_{\geq 1}$ is such that $a \equiv 3 \pmod 4$.
   (a) Let $S_n = 1 - \binom{n}{2}a + \binom{n}{4}a^2 - \binom{n}{6}a^3 + \cdots$. Show that $2S_n = (1+i\sqrt{a})^n + (1-i\sqrt{a})^n$.
   (b) Find a recurrence relation for S_n and show by induction that $2^{n-1} \mid S_n$. [Hint: you'll get a linear recurrence, remember from polynomials!]
4. In base b write the number
   $$x_n = \underbrace{11\ldots1}_{n-1}\underbrace{22\ldots2}_{n}5$$
   Suppose that for n large enough, x_n is a perfect square.
   (a) Show that $b-1$ is a perfect square. [Hint: Look at $b\sqrt{x_n} - \sqrt{x_{n+1}}$.]
   (b) (Harder) Show that $b = 10$.
5. Compute (and show the limit exists) the sequence
   $$1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}}$$
   [Hint: You have to be a little careful here, depending on the number of fractions you see.]
6. (Putnam 1993) Let $x_0, x_1, x_2, \ldots$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n+1}x_{n-1} = 1$ for all $n \geq 1$. Show that there exists a real a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

Harder

7. (Putnam 1990) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ for nonnegative m, n? [Hint: Yes.]
8. Suppose $k \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{R}$. Show that
   $$\lim_{n\to\infty} \binom{n}{k}\left(\frac{x}{n}\right)^k \left(1 - \frac{x}{n}\right)^{n-k} = \frac{x^k}{e^x k!}$$
9. Show that $\sqrt{1+\sqrt{2+\sqrt{3+\cdots}}}$ converges.
10. Let $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 6$ and
    $$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n$$
    Show that $n \mid a_n$ for $n \geq 1$. [Hint: Try some small cases and guess.]
11. Compute
    $$\lim_{n\to\infty} n^2 \int_0^{1/n} x^{x+1}dx$$
    [Hint: Put n^2 in the denominator and use limits of ratios.]

2.2 Series and products

Easier

12. Does

$$\sum_{n \geq 1} \ln\left(1 + \frac{1}{n}\right)$$

converge?

13. Does $\sum \sin(\pi\sqrt{n^2+1})$ converge? [Hint: Subtract off πn from the inside then use conjugates.]

14. Compute

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{n-1}+\sqrt{n}}$$

15. Compute $(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots$. [Hint: Multiply by $1-x$.]

16. (Putnam 1986) Evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \operatorname{arccot}(n^2+n+1)$.

Harder

17. (Putnam 2016) Let $x_0, x_1, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$, $x_{n+1} = \ln(e^{x_n} - x_n)$. Show that $\sum x_n$ converges and find its sum.

18. Suppose x_n are real such that $x_{n+1} \leq x_n + \frac{1}{n^2}$. Show that $\lim x_n$ exists. [Hint: Look at $x_n + 1 + 1/2^2 + \cdots + 1/(n-1)^2$.]

19. Compute the product

$$(1-4/1)(1-4/9)(1-4/25)\cdots$$