Math 43900 Problem Solving
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Lecture 11 Inequalities

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These problems are taken from the textbook, from Engels' *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Basics

Inequalities are a frequent and difficult topic on math competitions, and they are at the core of a huge number of results in analysis. Problem solving inequalities tend to be on the tricky side with ingenious algebra necessary to reduce them to some known inequalities. Nevertheless a handful of basic examples can be helpful in proving a large number of inequalities.

The basic inequalities:

1. By far the most useful inequality is that $x^2 \geq 0$ for all x real.
2. **AM-GM:** If $x_1, \ldots, x_n \geq 0$ then
$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$
with equality when $x_1 = x_2 = \ldots = x_n$.
3. **Cauchy-Schwarz:** If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are real numbers then
$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \geq (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2$$
with equality when $x_1 = \lambda y_1$, $x_2 = \lambda y_2$, $\ldots$, $x_n = \lambda y_n$ for a scalar λ.
4. **Chebyshev's inequality**: If $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$ then
$$x_1y_1+x_2y_2+\cdots+x_ny_n \geq x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+\cdots+x_ny_{\sigma(n)} \geq x_1y_n+x_2y_{n-1}+\cdots+x_ny_1$$

for any permutation σ.

Needless to say you may use any method from calculus to show inequalities, from minimization/maximization to Lagrange multipliers. Typically, however, reducing inequalities to the basic ones via algebraic manipulations is the most effective strategy. Brute force methods sometimes work, but they are very laborious.

Inequalities come is lots of guises but the following are major themes in problem solving:

1. Inequalities based on AM-GM
2. Inequalities based on Cauchy-Schwarz
3. Inequalities in geometry, where a useful fact is the triangle inequality.
4. Inequalities in calculus

2 Problems

2.1 AM-GM, Completing the square

Easier

1. Show that for all real numbers x
$$2^x + 3^x - 4^x + 6^x - 9^x \leq 1$$
[Hint: Complete the square.]
2. Show that $x^4 + 4x + 3 \geq 0$ for all real x. Find all positive integers n such that the equation
$$nx^4 + 4x + 3 = 0$$
has a real root.

Harder

3. Suppose $x_1, \ldots, x_n \in (1/4, 1)$. Show that
$$\log_{x_1}(x_2 - 1/4) + \log_{x_2}(x_3 - 1/4) + \cdots + \log_{x_n}(x_1 - 1/4) \geq 2n$$
[Hint: Show that $x^2 \geq x - 1/4$ and then use AM-GM.]

4. Suppose $a_1, \ldots, a_n$ are real numbers such that $a_1 + \cdots + a_n \geq n^2$ and $a_1^2 + \cdots + a_n^2 \leq n^3 + 1$. Show that $a_1, \ldots, a_n \in [n-1, n+1]$. [Hint: Enough to show that $a_k - n \in [-1, 1]$, or equivalently that $(a_k - n)^2 \leq 1$.]

5. Consider the real numbers $x_0 > x_1 > x_2 > \cdots > x_n$. Show that

$$x_0 + \frac{1}{x_0 - x_1} + \frac{1}{x_1 - x_2} + \cdots + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n$$

[Hint: Write $a_k = x_k - x_{k-1}$ and rewrite the inequality in terms of the a_k.]

2.2 Cauchy-Schwarz, Chebyshev

Easier

6. Find the maximum of the function $f(x, y, z) = 5x - 6y + 7z$ on the ellipsoid $2x^2 + 3y^2 + 4z^2 \leq 1$. [Hint: Use calc 3 if you're up for it, but it's much easier with Cauchy-Schwarz. For the latter, maximize $f(x, y, z)^2$.]

7. If $a_1 + a_2 + \cdots + a_n = n$ show that $a_1^4 + a_2^4 + \cdots + a_n^4 \geq n$. [Hint: Apply Cauchy-Schwarz twice.]

8. Show that the positive real numbers $a_0, a_1, \ldots, a_n$ form a geometric progression if and only if

$$(a_0 a_1 + a_1 a_2 + \cdots + a_{n-1} a_n)^2 = (a_0^2 + a_1^2 + \cdots + a_{n-1}^2)(a_1^2 + a_2^2 + \cdots + a_n^2)$$

Harder

9. Show that if $0 < a, b < \pi/2$ then

$$\frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b} \geq \sec(a - b)$$

10. Find the positive integers $n, k_1, \ldots, k_n$ are positive integers such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1$$

[Hint: Apply Cauchy-Schwarz to find n. Then play around.]

2.3 Inequalities in calculus and geometry

Easier

11. Suppose $f, g : [0,1] \to \mathbb{R}$ are continuous functions. Show that

$$\int_0^1 f^2(x)dx \int_0^1 g^2(x)dx \geq \left(\int_0^1 f(x)g(x)dx\right)^2$$

[Hint: Use Riemann sums and Cauchy-Schwarz.]

Harder

12. Show that in a triangle with sides a, b, c and area A one has

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}A$$

[Hint: $A = \frac{1}{2}bc \sin A$ and $a^2 = b^2 + c^2 - 2bc \cos A$.]

2.4 Solving equations using inequalities

Sometimes inequalities can be extremely effective in solving equations by eliminating large classes of potential solutions.

Easier

13. (Virginia Tech 2017) Find all nonnegative integers m and n such that $m^2 + 2 \cdot 3^n = m(2^{n+1} - 1)$. [Hint: You may use the following standard result from honors algebra 3: if $3^k \mid 2^n - 1$ then $2 \cdot 3^{k-1} = \varphi(3^k) \mid n$. Put in abstract algebra language: $(\mathbb{Z}/3^k\mathbb{Z})^\times$ is a cyclic group of order $\varphi(3^k)$ and 2 is a generator. To show this last statement show by induction that $2^{3^t} \equiv -1 + 3^{r+1} \pmod{3^{r+2}}$ and $4^{3^r} \equiv 1 + 3^{r+1} \pmod{3^{r+2}}$.]

Harder

14. (Romanian National 1999) Find x, y real such that

$$\begin{cases} 4^{-x} + 27^{-y} = \frac{5}{6} \\ 27^y - 4^x \leq 1 \\ \log_{27} y - \log_4 x \geq \frac{1}{6}. \end{cases}$$