Math 43900 Problem Solving
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Lecture 12 Functions and functional equations

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These problems are taken from the textbook, from Engels' *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Functions and functional equations

You've seen in physics and calculus differential equations where you were supposed to determine a particular function $f(x)$ satisfying a particular equation involing differentials. These are special examples of "functional equations", i.e., problems where you were supposed to determine a particular function $f(x)$ given only an equation satisfied by $f(x)$. They are a popular topic in math contests and solving them requires ingenuity and playfulness.

Example 1 (Cauchy's functional equation)**.** The most classical example of a simple (nondifferential) functional equation is to determine functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f(x + y) = f(x) + f(y)$$

As it stands the example has countless solutions (qnd I mean it in a technical way, there are uncountably many solutions). However, assuming mild properties of $f(x)$ one can show that $f(x) = ax$ for a fixed $a \in \mathbb{R}$ are the only solutions. This is the case when $f(x)$ is assumed to be continuous, or even integrable.

Remark 1. A large number of functional equations can be reduced to Cauchy's functional equation via alegbraic manipulations.

I identified 3 main topics:

1. Functional equations with integers, where you use the fact that the integers are discrete.
2. Functional equations over $\mathbb{R}$ where you use algebraic manipulations.
3. Functional equations over $\mathbb{R}$ where you use analytic properties of $f(x)$, such that continuity or differentiability or integrability.

2 Problems

2.1 Functional equations and the integers

Easier

1. Suppose $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfies $f(f(n)) = n + 3$ for all $n \geq 0$ integer.
   (a) Show that $f(n+3) = f(n) + 3$.
   (b) Deduce that $f(3k) = 3k + f(0)$, $f(3k+1) = 3k + f(1)$ and $f(3k+2) = 3k + f(2)$ for all nonnegative integers k.
2. Suppose $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfies $f(xf(y)) = \frac{f(x)}{y}$ for all $x, y \in \mathbb{Q}_{>0}$.
   (a) Show that $f(f(y)) = f(1)/y$, that $f(f(1)) = 1$ and deduce that $f(1) = 1$.
   (b) Deduce that $f(f(y)) = 1/y$ and show that $f(1/y) = 1/f(y)$. [Hint: Apply f to the first equation.]
3. Suppose $f : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ satisfies $f(n+1) > f(f(n))$ for all $n \geq 1$.
   (a) Show that $f(1)$ is the minimum value of f.
   (b) Show that $f(1) < f(2) < f(3) < \ldots$.

Harder

4. (Continuation of Exercise 1)
   (c) Show that $f(f(n)) \equiv n \pmod 3$ and conclude that either $f(x) \equiv x \pmod 3$ for at least one of $x \in \{0, 1, 2\}$.
   (d) Deduce that no such function $f(n)$ exists. [Hint: Use the previous part.]

5. (Continuation of Exercise 2)
   (c) Show that $f(x/y) = f(x)/f(y)$. [Hint: You know that $f(f(y)) = 1/y$.]
   (d) Deduce that $f(xy) = f(x)f(y)$ for all x, y.
   (e) Can you find ONE example of such f.

6. (Continuation of Exercise 3)
   (c) Show that $f(n) > n$ can never happen.
   (d) Deduce that $f(n) = n$ for all n.

2.2 Functional equations and algebraic manipulations

Easier

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 1/2$ and there is some real α for which
$$f(x+y) = f(x)f(\alpha - y) + f(y)f(\alpha - x)$$
for all $x, y \in \mathbb{R}$.
   (a) Show that $f(\alpha) = 1/2$.
   (b) Show that $f(\alpha - x) = f(x)$ for all x.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $xf(y) + yf(x) = (x+y)f(x)f(y)$. Show that for every $x \in \mathbb{R}$ we have $f(x) \in \{0, 1\}$. Can you show that f is an even function? [Hint: Play around with special values of x and y.]

9. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x)f(y) = f(x - y)$ for all x, y and also suppose that f is not the 0 function. Show that $f(0) = 1$ and that for every $x \in \mathbb{R}$, $f(x) \in \{-1, 1\}$. [Hint: Play around with special values of x and y.]

Harder

10. (Continuation of Exercise 7)
   (c) Show that $f(x) = \pm 1/2$ for all x.
   (d) Show that in fact $f(x) = 1/2$ for all x.

11. Determine all functions $f : [0, \infty) \to [0, \infty)$ satisfying the following properties: (a) $f(2) = 0$, (b) if $x \in [0, 2)$ then $f(x) \neq 0$ and (c) if $x, y \in [0, \infty)$ then $f(x + y) = f(xf(y))f(y)$.
12. Find the polynomials $P(X)$ such that $P(X + 1) = P(X) + 2X + 1$.
13. (Putnam 2016) Find all functions $f : (1, \infty) \to (1, \infty)$ with the following property: if $x, y \in (1, \infty)$ and $x^2 \leq y \leq x^3$ then $(f(x))^2 \leq f(y) \leq (f(x))^3$.

2.3 Functional equations and calculus

Easier

14. For each of the following functional equations find $f(x)$ continuous that satisfy the equation:
    (a) $f(x + y) = f(x)f(y)$ with $f : \mathbb{R} \to (0, \infty)$. [Hint: Use log.]
    (b) $f(x + y) = f(x) + f(y) + f(x)f(y)$. [Hint: Reduce case (a).]
    (c) $f(xy) = f(x) + f(y)$ for $f : (0, \infty) \to \mathbb{R}$.
    (d) $f(xy) = xf(y) + yf(x)$ for $f : (0, \infty) \to \mathbb{R}$. [Hint: Divide by xy.]

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15. Determine the continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$. [Hint: Can you reduce to Exercise 14 (a)?]
16. Find the continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation
$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}$$
[Hint: Compute $f(x/2)$ in terms of $f(x)$ and find a different functional equation satisfied by f.]
17. Determine the continuous functions $f : \mathbb{R} \to \mathbb{R}_{\neq 0}$ such that for all x, y
$$f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)}$$