Math 43900 Problem Solving
Fall 2017
Lecture 13 Brainstorming

Andrei Jorza

In previous lectures we concentrated on different problem solving topics and techniques, as well as writing up clearly and concisely your solutions. Today we'll concentrate on brainstorming, by which I mean how to poke around effectively when stuck while keeping a clear eye.

1 Problems

1. Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes such that the sum of the numbers in each box is the same? E.g., when $n = 8$ the example $(1, 2, 3, 6)$, $(4, 8)$, $(5, 7)$ shows that the largest k is at least 3.
2. Is there an infinite sequence of real numbers $a_1, a_2, \ldots$ such that for every positive integer m one has
$$a_1^m + a_2^m + \cdots = m?$$
3. Let $\mathcal{S}$ be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:
   (a) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in $\mathcal{S}$;
   (b) If $f(x), g(x)$ are in $\mathcal{S}$ then so are the function $f(x) + g(x)$ and $f(g(x))$;
   (c) If $f(x), g(x)$ are in $\mathcal{S}$ and $f(x) \geq g(x)$ for all $x \geq 0$ then the function $f(x) - g(x)$ is in $\mathcal{S}$. Prove that if $f(x), g(x)$ are in $\mathcal{S}$ then so is the function $f(x)g(x)$.
4. Let A be the $n \times n$ matrix whose entry on row i and column j is $1/\min(i, j)$. Compute $\det A$.
5. (Putnam 1960) Consider the sequence $(a_n)_{n\geq 0}$ defined by $a_0 = 0$ and $a_{n+1} = 1 + \sin(a_n - 1)$ for $n \geq 0$. Compute
$$\lim_{n\to\infty} \frac{a_0 + a_1 + \cdots + a_n}{n}.$$
6. (Putnam 1961) The set of pairs of positive reals (x, y) such that $x^y = y^x$ form the straight line $y = x$ and a curve. Find the point at which the curve cuts the line.
7. (Putnam 1963) Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x)^2 - f(y)^2 = f(x + y)f(x - y),$$
for all reals x, y.
8. (Putnam 1968) Let S be a finite set and P be the set of all subsets of S. Show that we can label $P = \{A_0, A_1, \ldots\}$ such that $A_0 = \emptyset$ and for each $n \geq 1$, either $A_{n-1} \subset A_n$ with $|A_n - A_{n-1}| = 1$ or $A_n \subset A_{n-1}$ with $|A_{n-1} - A_n| = 1$.
9. (Putnam 1967) Find the smallest positive integer n such that we can find a polynomial $nx^2 + ax + b$ with integer coefficients and two distinct roots in the interval $(0, 1)$.