Math 43900 Problem Solving
Fall 2017
Lecture 14 Which problems do I choose?

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These problems are taken from older Putnams.

1 What?

One of the most difficult issues with problem solving competitions, much like problem solving in the real world, is to know which problems to try, and when to give up when stuck? This is something that can only be learned from experience. It's not enough to know the rules, you need to practice it to be good at it. For example, on this year's VTRMC I worked on the problems in the following order: 1, 2, 1, 2, 3, 4, 1, 5, 2, 6, 4, 3, 7, 5, 3, 7.

2 Problems

1. Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets satisfying the property $A_1 \cup A_2 \cup A_3 = \{1, 2, \ldots, 10\}$ and $A_1 \cap A_2 \cap A_3 = \emptyset$. Express your answer in the form $2^a 3^b 5^c 7^d$. (For a simpler version replace the condition $A_1 \cap A_2 \cap A_3$ with A_1, A_2, A_3 are pairwise disjoint.)
2. Find the maximum of $f(x) = x^3 - 3x$ on the region $\{x \mid x^4 + 36 \leq 13x^2\}$.
3. Inscribe a rectangle of base b and height h in a circle of radius 1, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the isosceles triangle). For what value of h do the rectangle and the triangle have the same area?
4. Let R be the region consisting of points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.
5. How many primes among the positive integers, written as usual in base 10, are of the form $10101 \ldots 10101$, i.e., their digits are alternating 1s and 0s, beginning and ending with 1.

6. Evaluate $\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2)} dydx$, where $a, b > 0$.

7. A 2×3 rectangle has vertices at $(0,0), (2,0), (0,3)$, and $(2,3)$. It rotates 90° clockwise about the point $(2,0)$. It then rotates 90° clockwise about the point $(5,0)$, then 90° clockwise about the point $(7,0)$, and finally, 90° clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is $(1,1)$.

8. Prove that $f(n) = 1 - n$ is the only integer valued function defined on the integers that satisfies the following conditions:

   (a) $f(f(n)) = n$ for all n,

   (b) $f(f(n+2)+2) = n$ for all n,

   (c) $f(0) = 1$.

9. Evaluate $\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} dx$.

10. Show that every composite integers can be written as $xy + yz + zx + 1$ for some positive integers x, y, z.

11. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer as $(a\sqrt{b} + c)/d$ with integers a, b, c, d.

12. Find all real-valued continuous differentiable functions on the real line such that for all real x $(f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2)dt + 1990$.

13. Find all positive integers that are within 250 of exactly 15 perfect squares.

14. For a partition π of $\{1, 2, \ldots, 9\}$ let $\pi(x)$ be the number of elements in the part containing x. Prove that for every two partitions π and π', there exist two distinct integers x and y in $\{1, 2, \ldots, 9\}$ such that $\pi(x) = \pi'(x)$ and $\pi(y) = \pi'(y)$.

Hints

1. Interpret the problem as asking the number of ways of placing the numbers 1 through 10 in a Venn diagram.
2. Factor $x^4 - 13x^2 + 36$.
3. Express the altitude in terms of h.
4. Graph the part of R in the first quadrant. Use symmetry.
5. Note that $100^k - 1 = (10^k - 1)(10^k + 1)$.
6. Divide the rectangle into two parts by the diagonal $ay = bx$.
7. The region is a disjoint union of triangles and quarter circles.
8. Apply f to (b) and then use (a) on the LHS.
9. Use the symmetry of the interval.
10. Take $z = 1$.
11. Assume that the dartboard has corners $(\pm 1, \pm 1)$ and find the equations of the curves bounding one-eigth of the specified region.
12. If g, h are differentiable and $g(0) = h(0)$ and $g' = h'$ then $g = h$.
13. The 15 squares must be consecutive.
14. For any partition π, $\pi(x)$ can take at most 3 values.