Math 43900 Problem Solving
Fall 2017
Lecture 2 Exercises

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These problems are taken from the textbook, from past Putnam competitions, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

Proof by contradiction

1. Show that the equation $3^y + 5^z = x^2 + x + 1$ has no integer solutions.
2. Three points A, B and C lie on a circle (ellipse). Show that the area of ABC is maximized if and only if its centroid (i.e., its center of mass, or intersection of its three medians) coincides with the center of the circle (ellipse). (Putnam 1952)
3. A polynomial $P(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ with integral coefficients has a_n, a_0 and $P(1)$ all odd integers. Show that $P(X)$ has no rational roots. (Putnam 1952)
4. The set $\mathcal{S} \subset \mathbb{Q}$ satisfies:
   (a) $\mathcal{S}$ is closed under addition and multiplication,
   (b) $0 \notin \mathcal{S}$, and
   (c) for any nonzero rational $q \in \mathbb{Q}$ exactly one of q and $-q$ belongs to $\mathcal{S}$.

   Show that $\mathcal{S} = \mathbb{Q}_{>0}$.

Mathematical induction

Induction where you know what you need to show

1. Show that a $2^n \times 2^n$ board with one unit square removed can be tiled with corner tiles (a corner tiles has 3 unit square in an L shape).
2. Show that $3^n \geq n^3$ for all positive integers n. (AG 14)
3. Let $n \geq 6$ be an integer. Show that
$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$
   (AG 15)
4. Prove that any polygon (convex or not) can be dissected into triangles by interior diagonals. (AG 29)
5. Show that for any $n \geq 4$ an isosceles triangle with one angle of 120° can be dissected into n triangles similar to it. (AG 26)
6. Show that every positive integer can be written in the form $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some $n \geq 1$ and some choice of signs.

Due next week

Write

Please write out clearly and concisely one of the following:

1. one problem from the ones I explained in class and one problem of your choosing that I did not cover in class OR
2. two problems that I did not cover in class.

Read

In preparation for next class, please read from the textbook sections 1.2 (induction) and 1.3 (the pigeonhole principle). I encourage you to work out on paper the examples the textbook presents: be an active participant in the presentation.