Math 43900 Problem Solving
Fall 2017
Lecture 3 Exercises

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These problems are taken from the textbook, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

Mathematical induction

Induction where you know what you need to show

1. Prove that for any positive integer $n \geq 2$ there exists a positive integer m that can be written simultaneously as a sum of $2, 3, \ldots, n$ squares of nonzero integers.

2. Show by induction that for $n \geq 1$, 2^{n+3} divides $3^{2^n} - 2^{n+2} - 1$. A slightly more general result, but with the same proof. Suppose a is an odd number and ℓ is a positive integer such that $a - 1 - 2^\ell$ is a multiple of $2^{\ell+1}$. Show that for all $n \geq 0$, $2^{\ell+n+1}$ divides $a^{2^n} - 1 - 2^{\ell+n}$. (The former is obtained by applying the latter to $a = 3^2$.)

3. Let a_n be the number of ways to tile a $1 \times n$ strip by 1×1 and 1×3 tiles. Show that $a_n < 1.5^n$.

4. Show that if $a_1, \ldots, a_n > 0$ then
$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq (1 + \sqrt[n]{a_1 \cdots a_n})^n.$$

Induction where you need to figure out what you want to prove

If you don't know what precise statement to prove by induction, you should try some small cases to guess the statement you'd like to prove.

1. Find a formula for the sum of the first n odd numbers.

2. Show that for all positive integers n
$$1 + \frac{1}{2^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}$$
[As it stands this looks hard to tackle by induction. Amusingly, the slightly harder inequality where you replace $< \frac{3}{2}$ with $< \frac{3}{2} - \frac{1}{n}$ can be done with induction. What is the base case?]

3. Find a formula for x_n knowing that $x_1 = \frac{5}{2}$ and $x_{n+1} = x_n^2 - 2$ for all $n \geq 1$.

4. Find a closed formula for $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}^n$. (Useful for differential equations!)

The pigeonhole principle

1. Show that at any party there are two people who know exactly the same number of people at the party.
2. Consider integers $1 \leq a_1 < a_2 < \ldots < a_{50} < 100$. Show that $a_i + a_j = 99$ for some i and j.
3. Inside a circle of radius 4 are 45 points. Show that you can find two of these points at most $\sqrt{2}$ apart. [Hint: Draw circles around each point.]
4. Given n integers, prove that some nonempty subset of them has sum divisible by n.
5. (Erdös) Let $A \subset \{1, 2, \ldots, 2n\}$ be a set of $n+1$ integers. Prove that some element of A divides another.
6. (Putnam 2002). Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.
7. (Putnam 2000). Let a_j, b_j, c_j be integers for $1 \leq j \leq n$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $\frac{4}{7}n$ values of j between 1 and n.
8. (Putnam 1994) Let A and B be 2 by 2 matrices with integer entries such that $A, A+B, A+2B, A+3B$ and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.
9. (IMO 1972) Prove that from a set of ten distinct two-digit numbers, it is possible to select two non-empty disjoint subsets whose members have the same sum.

Due next week

Write

Please write out clearly and concisely one of the following:

1. one problem from the ones I explained in class and one problem of your choosing that I did not cover in class OR
2. two problems that I did not cover in class.

Read

In preparation for next class, please read from the textbook section 2.2 (polynomials). I encourage you to work out on paper some of the examples the textbook presents: be an active participant in the presentation.

Attempt

Please look over the problems from the following lecture. This way you can ask me questions and we can discuss solutions in class.

Stats

I ran some stats on Putnam problems and saw that about a quarter of the problems are combinatorics, a quarter are number theory. The next batch is geometry, algebra, analysis and inequalities with about 15% each. The third batch, with about 10% each, consists of functions, sequences, polynomials, matrices. Other nontrivial categories are derivatives, integrals, limits, equations, series, probabilities, abstract algebra.

Since most problems belong to more than one category, and it's sometimes not clear what category a problem belongs to, these stats are purely qualitative.