# Math 43900 Problem Solving
# Fall 2017
# Lecture 6 Invariants

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These problems are taken from the textbook, from Engels' *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

## The Idea

Often one has to show that a particular configuration is not possible, or that a configuration cannot be obtained from another configuration via certain types of changes. The idea is to attach to a configuration an **invariant** or a **semi-invariant**. The invariant stays the same while the semi-invariant keeps increasing (or decreasing). How do such problems work? To show a configuration is not possible or is not attainable you show that its invariant or semi-invariant is of the wrong type.

## Invariants

### Easier

1. Show that a $6 \times 6$ board cannot be covered with $4 \times 1$ pieces. What about a $2006 \times 2006$ board? What about an $n \times n$ board?

2. If you remove opposite corners of a $10 \times 10$ board, is it possible to cover the rest with 49 dominoes (of size $2 \times 1$)?

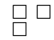
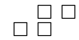
3. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times? [Hint: try to find some expression that stays the same after each move.]

4. Consider the polynomials $P(X) = X^2 + X$ and $Q(X) = X^2 + 2$. Starting with the list $\{P(X), Q(X)\}$. You may keep increasing the list as follows: take any two polynomials $f$ and $g$ in the list, and add to the list $f + g$ or $f - g$ or $fg$. Is it possible that after finitely many such steps the list contains the polynomial $X$?

### Harder

5. Start with the number $7^{2016}$. At every step you erase the first digit and add it to the remaining number. (E.g., 1234 is replaced by $234 + 1 = 235$.) You stop when you arrive at a 10 digit number. Show that this number has two equal digits. [Hint: think, among others, of pigeonhole.]

6. You are given an ordered triple of numbers. You are allowed to choose any two of them, say $a$ and $b$ and replace them by $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. If you start with the triple $(1, \sqrt{2}, 1+\sqrt{2})$ can you get to the triple $(2, \sqrt{2}, 1/\sqrt{2})$ via a finite number of such changes? [Hint: Play around in the plane first.]

7. (Putnam 2016) Let $m, n \geq 4$. A $(2m-1) \times (2n-1)$ board is covered with  and . What's the smallest number of tiles you need?

# Semi-invariants

## Easier

8. A real number is written in each square of an $n \times n$ chessboard. We can perform the operation of changing all signs of the numbers in a row or a column. Prove that by performing this operation a finite number of times we can produce a new table for which the sum of each row or column is positive.

9. Nine of the unit cells on a $10 \times 10$ board are infected. Every minute, the cells with at least 2 infected neighbors become infected. Show that there is always an uninfected cell. [Hint: Look at the perimeter of the infected squares.]

10. $n$ positive numbers are written on a board. In a step you may erase any two of these numbers, say $a$ and $b$, and write instead $(a+b)/4$. Repeating this step $n-1$ times there is only one number left on the board. Show that this number is at least $1/n$. [Hint: Look at the sum of reciprocals of the numbers on the board.]

## Harder

11. Suppose you have real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$. Show that for every permutation $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ of the indices $\{1, 2, \ldots, n\}$ one has

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \leq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_n y_{\sigma(n)} \leq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

[Hint: When $i < j$ but $a > b$ what happens when you replace $x_i y_a + x_j y_b$ by $x_i y_b + x_j y_a$?]

12. $N$ men and $N$ women are distributed among the rooms of a mansion. They move among the rooms according to the rules: either

    (a) a man moves from a room with more men than women (counted before he moves) into a room with more women than men, or

    (b) a woman moves from a room with more women than men into a room with more men than women.

    Show that eventually people will stop moving. [Hint: try to find some expression that keeps decreasing after each move.]