Math 43900 Problem Solving
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Lecture 9 Matrices

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These problems are taken from the textbook, from Engels' *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Matrices

Overview

The way matrices show up in problem solving problems involves the following three main themes:

1. algebraic manipulations of matrices (they can be multiplied and the operation is not commutative),
2. determinants and eigenvalues of matrices,
3. matrices as defining linear maps on vector spaces.

Basic results

1. You can always add two $m \times n$ matrices.
2. You can always multiply an $m \times n$ matrix and an $n \times p$ matrix to get an $m \times p$ matrix.
3. The **trace** of a matrix $\operatorname{Tr} A$ is the sum of its diagonal terms. It has the property that $\operatorname{Tr}(A+B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$ and $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for all matrices A and B.
4. The **determinant** of a matrix $\det A$ is a polynomial expression in the entries of the matrix A and satisfies the following properties:
   (a) The determinant of (a_{ij}) is $\displaystyle\sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$, where S_n is the group of permutations and $\varepsilon(\sigma)$ is the sign. The sign ε is multiplicative and if τ is a k-cycle then $\varepsilon(\tau) = (-1)^{k-1}$.
   (b) If in a matrix $A = (a_{ij})$ you write $A_{p,q}$ for the $(n-1) \times (n-1)$ where you eliminate the p-th row and q-th column from A then
   $$\det(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n-1} a_{1,n} \det A_{1,n}$$
   (c) $\det(AB) = \det(A)\det(B)$ for all matrices A and B.
   (d) If you swap two rows or columns of a matrix A to obtain a matrix B then $\det(B) = -\det(A)$.
   (e) If in a matrix A you add a multiple of one row to a different row to get a matrix B then $\det(B) = \det(A)$. The same is true if you add a multiple of a column to a different column.

5. Suppose A is an $n \times n$ matrix. If you can find a **nonzero** vector (i.e., an $n \times 1$ matrix consisting of a single column) and a scalar α such that $Av = \alpha v$ then α is said to be an eigenvalue of A with eigenvector v.

6. If A is an $n \times n$ matrix the **characteristic polynomial** of A is the monic degree n polynomial

$$P_A(X) = \det(XI_n - A)$$

   (a) A scalar α is an eigenvalue of A if and only if it is a root of $P_A(X)$. The roots of $P_A(X)$ are **the** eigenvalues of A and are counted with multiplicity if they are not distinct. E.g., I_n has n eigenvalues all equal to 1.

   (b) $P_A(X) = X^n - (\operatorname{Tr} A)X^{n-1} + \cdots + (-1)^n \det(A)$.

   (c) Since we know the relation between the coefficients of a polynomial and its roots we deduce that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A then

$$\begin{aligned} \operatorname{Tr}(A) &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \\ \det(A) &= \lambda_1 \lambda_2 \cdots \lambda_n \end{aligned}$$

   (d) If you plug in A into the polynomial $P_A(X)$ you always get the 0 matrix: $P_A(A) = O$.

   (e) If A and B are matrices then $P_{AB}(X) = P_{BA}(X)$ as polynomials.

7. A big result in linear algebra says that for any matrix A you can find an invertible matrix S such that the conjugate SAS^{-1} has a very special shape: the **Jordan canonical form**. In fact the Jordan canonical form SAS^{-1} has the n eigenvalues on the diagonal but much more is true: SAS^{-1} is block diagonal and each block is of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \ldots \\ 0 & \lambda & 1 & \ldots \\ & & \ddots & \ddots \\ 0 & \ldots & 0 & \lambda \end{pmatrix}$$

with an eigenvalue λ on the diagonal and 1-s off diagonal. E.g., for a 2×2 matrix the possible Jordan canonical forms are

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ for } \lambda_1 \neq \lambda_2 \text{ and } \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

2 Problems

2.1 Determinants, traces, characteristic polynomials and eigenvalues

Easier

1. Show that you can never find two $n \times n$ matrices A and B with real coefficients such that $AB - BA = I_n$. [Hint: What is the trace?]

2. Consider an $n \times (n+1)$ matrix $A = (a_{ij})$. For a column k write A_k for the $n \times n$ matrix you obtain from A by removing the k-th column. Show that

$$a_{11} \det A_1 - a_{12} \det A_2 + \cdots + (-1)^{n+1} a_{1,n+1} \det A_{n+1} = 0$$

[Hint: Can you see this expression as the determinant of an $(n+1) \times (n+1)$ matrix?

3. Suppose $P(X)$ is a polynomial and A is an $n \times n$ matrix such that $P(A) = 0$. Show that the eigenvalues of A are among the roots of $P(X)$. [Hint: What are the eigenvalues of $P(A) = 0$? Use Exercise 8.]

4. This is an application of Exercise 19. Suppose X is an antisymmetric matrix, i.e., of the form $X = -X^t$. (Think $\begin{pmatrix} & x \\ -x & \end{pmatrix}$.) Show that every eigenvalue of X is of the form ai where $i = \sqrt{-1}$ and $a \in \mathbb{R}$. [Hint: If $Xv = \alpha v$ compute $\langle X^t v, v\rangle = \langle v, Xv\rangle$ in two ways.]

5. Show that $A^k = 0$ for some $k \geq 0$ if and only if all the eigenvalues of A are 0 in which case $A^n = 0$ as well.

6. (Putnam 1994) Let A and B be 2 by 2 matrices with integer entries such that $A, A+B, A+2B, A+3B$ and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

7. Let $p < m$ be positive integers. Show that

$$\det \begin{pmatrix} \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{p} \\ \binom{m+1}{0} & \binom{m+1}{1} & \cdots & \binom{m+1}{p} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m+p}{0} & \binom{m+p}{1} & \cdots & \binom{m+p}{p} \end{pmatrix} = 1.$$

Harder

8. (**VERY USEFUL**) Suppose A is an $n \times n$ matrix and $Q(X)$ is any polynomial. If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $Q(A)$ (also an $n \times n$ matrix) are $Q(\lambda_1), \ldots, Q(\lambda_n)$.

9. Suppose A is an $n \times n$ real matrix such that $A^2 = A + I_n$. Show that $\det(A) < 2^n$. In fact show that $\det(A) \leq \left(\frac{1+\sqrt{5}}{2}\right)^n$.

10. Suppose X is a real matrix with $X + X^t = I_n$. Show that $\det X \geq \frac{1}{2^n}$.

11. Compute the determinant of the matrix (a_{ij}) where $a_{ii} = 2$ and if $i \neq j$ then $a_{ij} = (-1)^{i-j}$. [Hint: Use row operations to simplify the matrix.]

12. Let A and B be 3×3 matrices with real elements such that $\det A = \det B = \det(A \pm B) = 0$. Show that $\det(xA + yB) = 0$ for all real numbers x, y.

13. Let n be an odd positive integer. Suppose A is an $n \times n$ matrix whose square A^2 is either 0 or I_n. Show that $\det(A + I_n) \geq \det(A - I_n)$.

14. Suppose A and B are commuting $n \times n$ matrices with real entries such that $\det(A + B) \geq 0$. Show that $\det(A^k + B^k) \geq 0$ for all $k \geq 1$.

2.2 Algebraic operations and linear algebra

Easier

15. Suppose (x_n) is a sequence defined by the linear recurrence $x_{n+2} = ax_{n+1} + bx_n$ for all $n \geq 0$. Show that

$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$$

and conclude that for $n \geq 1$, x_n is the first entry of the matrix $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$.

16. Compute $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n$ and $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n$ for all n.

17. Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is a converging power series. Show that $f(SAS^{-1}) = Sf(A)S^{-1}$.

18. A useful application of Exercise 16. Show that if $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is an absolutely convergent power series then $f\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right) = \begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix}$ and $f\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = \begin{pmatrix} f(\lambda) & f'(\lambda) \\ 0 & f(\lambda) \end{pmatrix}$.

19. If u and v are $n \times 1$ column matrices write $\langle u, v \rangle = u^t v$ for the dot product of the two vectors. If A is an $n \times n$ matrix show that $\langle u, Av \rangle = \langle A^t u, v \rangle$. Show that $\langle v, \overline{v} \rangle \geq 0$, where $\overline{v}$ is the complex conjugate of v.

20. If $A = (a_{ij})$ show that $\mathrm{Tr}(A \cdot A^t) = \sum_{i,j} a_{ij}^2$.

Harder

21. Show that there exists no complex matrix A such that $\sin(A) = \begin{pmatrix} 1 & 2016 \\ 0 & 1 \end{pmatrix}$. (This is a Putnam problem.) [Hint: Use the Exercise 18.]

22. Suppose A and B are 2×2 complex matrices such that $AB = BA$. Show that you can find two complex numbers a and b such that $B = aA + b$. [Hint: Conjugate A to a Jordan canonical form, then things are much easier.] More generally, if A and B are $n \times n$ matrices such that $AB = BA$ show that $B = P(A)$ where P is a degree at most $n - 1$ polynomial.

23. Consider $v_1, \ldots, v_n$ vectors in $\mathbb{R}^n$ and the matrix A whose entry on row i and column j is the dot product $v_i \cdot v_j$. Let B be the matrix whose columns are the vectors $v_1, \ldots, v_n$. Show that $\det A = (\det B)^2$. [Hint: Show that $A = B^t B$.]

24. An application of the previous problem. Suppose $A_1, A_2, \ldots, A_n$ are subsets of some set $\{x_1, x_2, \ldots, x_n\}$. Show that if M is the $n \times n$ matrix whose entry on row i and column j is the cardinality $|A_i \cap A_j|$ then $\det M \geq 0$. [Hint: Apply the previous problem to v_i whose entry in position j is 1 if x_j is in A_i and 0 otherwise.] A cool application: Suppose you choose $a_1, \ldots, a_n$ divisors of an integer n. Show that $\det(\gcd(a_i, a_j)) \geq 0$.

25. Suppose A and B are two $n \times n$ matrices that don't commute and you can find nonzero real numbers p, q, r such that $pAB + qBA = I_n$ and $A^2 = rB^2$. Show that $p = q$.

26. Suppose A and B are $n \times n$ real matrices such that $\mathrm{Tr}(A \cdot A^t + B \cdot B^t) = \mathrm{Tr}(A \cdot B + A^t \cdot B^t)$. Show that $A = B^t$. [Hint: Use Exercise 20 and then complete squares.]