Mock Putnam

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For next week please set aside 2 consecutive hours and works on these exercises as if they were a real competition. Please submit your solutions but not your scrap paper, as you would in a real competition. If you do not know how to solve a problem, know that I have been to competition where I couldn't solve any problem myself. In that case, if you have an idea, please write that idea clearly and work out some test cases. You can submit that, you never know, you might happen on a good idea.

Things to keep in mind:

- You don't need to try every problem.
- You can give up when you're stuck and try another one.
- You can start with whichever problem sounds better to you.

1. (Putnam 1963 B1) Find all integers n for which $x^2 - x + n$ divides $x^{13} + x + 90$ in $\mathbb{Z}[x]$.
2. (Putnam 1962 A2) Find all integrable functions $f : [0, \infty) \to \mathbb{R}$ such that
$$\left(\frac{1}{x} \int_0^x f(t)dt \right)^2 = f(0)f(x),$$
for all $x \geq 0$.
3. (Putnam 1966 B2) Prove that at least one integer in any set of ten consecutive integers is relatively prime to the others in the set.
4. (Putnam 1969 A2) Let A be an $n \times n$ matrix with elements $a_{i,j} = |i - j|$ for $1 \leq i, j \leq n$. Show that $\det A = (-1)^{n-1}(n-1)2^{n-2}$.
5. (Putnam 1980 A2) Find $f(m, n)$, the number of 4-tuples (a, b, c, d) of positive integers such that the lowest common multiple of any three integers in the 4-tuple is $3^m \cdot 7^n$.
6. (Putnam 1981 B2) What is the minimum value of
$$(a-1)^2 + \left(\frac{b}{a} - 1\right)^2 + \left(\frac{c}{b} - 1\right)^2 + \left(\frac{4}{c} - 1\right)^2$$
over all real numbers a, b, c satisfying $1 \leq a \leq b \leq c \leq 4$?