Math 30820 Honors Algebra 4
Homework 1

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Due Wednesday, 1/25/2017

Do 4 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1. Determine, with proof, the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$.

2. Determine, with proof, the minimal polynomial of $\sqrt{2+\sqrt{2+\sqrt{2}}}$ over $\mathbb{Q}$.

3. Determine, with proof, the minimal polynomial of the element $\sqrt{X+\sqrt[p]{X}}$ over the PID $\mathbb{F}_p[X]$. Here p is a prime and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

4. (Generalized Eisenstein criterion) Suppose R is a unique factorization domain and $\mathfrak{p}$ is a prime ideal of R such that $R/\mathfrak{p}$ is also a unique factorization domain. Let $P(X) \in R[X]$ be $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$. Show that if $a_0, \ldots, a_{n-1} \in \mathfrak{p}$ but $a_0 \notin \mathfrak{p}^2$ then $P(X)$ is irreducible in $R[X]$ (and therefore also in $(\operatorname{Frac} R)[X]$ by Gauss' lemma from last semester).

5. Suppose $R \subset S$ are rings. An element $\alpha \in S$ is said to be *integral over* R if $P(\alpha) = 0$ for some **monic** polynomial $P \in R[X]$. Suppose R is a unique factorization domain. Show that if $\alpha \in \operatorname{Frac} R$ is integral over R then $\alpha \in R$.

6. Suppose α is integral over a ring R. Show that $R[\alpha]$ (defined last semester as $\{P(\alpha) \mid P \in R[X]\}$) is in fact the set $\{a_0 + a_1\alpha + \cdots + a_n\alpha^n \mid a_0, \ldots, a_n \in R\}$ for some integer n.