Math 30820 Honors Algebra 4
Homework 3

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Do 6 of the following questions.
Throughout this problem set R is an **integral domain**, unless otherwise specified.

1. Let R be a ring, I an ideal of R and M an R-module.

   (a) Show that $IM = \{\sum_{\text{finite} a_i m_i | a_i \in I, m_i \in M\}}$ is an R-submodule of M.

   (b) Show that M/IM is an R/I-module.

   Proof. (a): IM is clearly closed under addition. If $r \in I$ and $\sum a_i m_i \in IM$ then $r \sum a_i m_i \in IM$ as $ra_i \in I$ because I is an ideal. Therefore IM is a submodule of M.

   (b): If $r+I \in R/I$ and $m+IM \in M/IM$ we define scalar multiplication by $(r+I)(m+IM) = rm+IM$. We only need to check that this is well defined. I.e., we need to check that if $r' + I = r + I$ and $m' + IM = m + IM$ then $rm + IM = r'm' + IM$. But then $r' = r + a$ for $a \in I$ and $m' = m + bn$ with $b \in I$ and $n \in M$. Therefore

   $$r'm' = (r + a)(m + bn) = rm + am' + brn$$

   and so $r'm' - rm \in IM$ as desired. □

2. Consider $\mathbb{C}$ as a $\mathbb{Z}[i]$-module under usual multiplication of complex numbers. Determine the torsion submodule of the $\mathbb{Z}[i]$-module $\mathbb{C}/\mathbb{Z}[i]$.

   Proof. $\text{Tor}(\mathbb{C}/\mathbb{Z}[i]) = \{z \in \mathbb{C} \mid (m + ni)z \in \mathbb{Z}[i] \text{ for some } m + ni \in \mathbb{Z}[i]\}$. Therefore the torsion module consists of $\text{Frac}(\mathbb{Z}[i])/\mathbb{Z}[i] = \mathbb{Q}[i]/\mathbb{Z}[i]$. □

3. Consider the ring $A = \mathbb{F}_2[X]/(X^2 - X)$ with 4 elements. Show that torsion elements of the free A-module A do not form a submodule. (Note that A is not a domain so this does not contradict the statement from class.)

   Proof. Write $R = \{0, 1, X, X - 1\}$. Clearly $X(X - 1) = 0$ but $X, X - 1 \neq 0$ so $X, X - 1 \in \text{Tor}(A)$. However, their difference is $1 \notin \text{Tor}(A)$. Therefore $\text{Tor}(A)$ is not closed under addition. □

4. Let M be a finitely generated R-module and $\{m_1, \ldots, m_n\}$ a linearly independent subset of M.

   (a) Show that $N = \langle m_1, \ldots, m_n \rangle$ is free $\cong R^n$.

   (b) If $\{m_1, \ldots, m_n\}$ is a maximal linearly independent subset show that M/N is torsion, i.e., every element of M/N is annihilated by a nonzero element of R.

Proof. (a): From class the map $R^n \to M$ given by $\phi((x_i)) = \sum x_i m_i$ is an R-module homomorphism with image N. As the m_i are linearly independent we deduce that $\ker \phi = 0$ so the 1st isomorphism theorem implies $N \cong R^n$.

(b): Choose any nonzero $m \in M/N$, i.e., $m \notin N$. By maximality of the linearly independent set $m_1, \ldots, m_n$ it follows that $m, m_1, \ldots, m_n$ are linearly dependent and so $am + \sum a_i m_i = 0$ for $a, a_i \in R$ not all 0. Note that $a \neq 0$ as the m_i are linearly independent. But then $am \in N$ so $am = 0$ in M/N with $a \neq 0$ so m is torsion in M/N. □

5. Let M be an R-module and $\operatorname{Tor}(M)$ its torsion submodule. Show that $M/\operatorname{Tor}(M)$ is torsion-free, i.e., $\operatorname{Tor}(M/\operatorname{Tor}(M)) = 0$.

Proof. If $m \in \operatorname{Tor}(M/\operatorname{Tor}(M))$ then for some nonzero $a \in R$, $am = 0$ in $M/\operatorname{Tor}(M)$, i.e., $am \in \operatorname{Tor}(M)$. By definition for some nonzero $b \in R$, $b(am) = 0$ so $(ab)m = 0$. Since $a, b \neq 0$ and R is a domain we get that $m \in \operatorname{Tor}(M)$ and so $m = 0$ in $M/\operatorname{Tor}(M)$. □

6. Consider the R-module $M = R^n \oplus N$ where N is a torsion module, i.e., $N = \operatorname{Tor}(N)$. Let $e_1, \ldots, e_n$ be the standard basis of R^n and $t_1, \ldots, t_n \in N$ arbitrary elements. Show that $v_1 = e_1 + t_1, \ldots, v_n = e_n + t_n$ are linearly independent and the map $f : M \to M$ defined as the identity on N and sending $e_i \mapsto v_i$ is an isomorphism. (The point of this exercise is that while $N = \operatorname{Tor}(M)$ is well-defined solely in terms of M, the free part R^n is not as every basis can be changed by torsion elements to get another basis.)

Proof. If $\sum a_i(e_i + t_i) = 0$ then $\sum a_i e_i = -\sum a_i t_i \in N$. By the torsion assumption on N for some nonzero $b \in R$ we have $b \sum a_i e_i = 0$. By linear independence of the e_i we get $ba_i = 0$ for all i. As R is a domain and $b \neq 0$ we get all $a_i = 0$.

Consider the map $f : M = R^n \oplus N \to M$ given by $f(r_1, \ldots, r_n, v) = \sum r_i v_i + n$. One can write this also as $f((r_i), n) = ((r_i), n + \sum r_i t_i)$. It is clearly an R-module homomorphism. Suppose $g : M \to M$ is the homomorphism $g((r_i), n) = ((r_i), n - \sum r_i t_i)$. Then $f \circ g = g \circ f = \mathrm{id}_M$ and so f and g as isomorphisms. □

7. Let $f : M \to N$ be a homomorphism of R-modules. If $\operatorname{Im} f$ and $\ker f$ are finitely generated, show that M is finitely generated.

Proof. Suppose $\ker f$ is generated by $k_1, \ldots, k_a$ and $\operatorname{Im} f$ is generated by $n_1 = f(m_1), \ldots, n_b = f(m_b)$. Let $m \in M$. Then $f(m) \in \operatorname{Im} f$ so for some $r_i \in R$ have $f(m) = \sum r_i n_i = \sum r_i f(m_i) = f(\sum r_i m_i)$.

But then $f(m - \sum r_i m_i) = 0$ so $m - \sum r_i m_i \in \ker f$ is generated by the k_j. We can therefore find $s_1, \ldots, s_a \in R$ such that $m - \sum r_i m_i = \sum s_j k_j$. This implies that M is generated by the k_j and the m_i. □

8. Let $\phi : R \to S$ be a ring homomorphism and M an S-module. For $r \in R$ and $m \in M$ define $r \cdot m := \phi(r)m$, the later being scalar multiplication in M by $\phi(r) \in S$.

   (a) Show that this operation yields an R-module structure on the abelian group M. Call $\phi^* M$ this R-module.

   (b) If $f : M \to N$ is a homomorphism of S-modules define $\phi^* f : \phi^* M \to \phi^* N$ by $\phi^* f(m) = f(m)$. Show that $\phi^* f \in \operatorname{Hom}_R(\phi^* M, \phi^* N)$.

Proof. (a): We need to check that if $r, r' \in R$ and $m, m' \in M$ then $(r + r') \cdot m = r \cdot m + r' \cdot m$ and $r \cdot (m + m') = r \cdot m + r \cdot m'$. This follows from the fast that ϕ is additive. Also we need to check that $(rr') \cdot m = r \cdot (r' \cdot m)$ which again is immediate as ϕ is multiplicative.

(b): The map f is already a homomorphism of abelian groups (and as abelian group M is the same as $\phi^* M$) so we only need to check R-linearity. We need to check that $\phi^* f(r \cdot m) = r \cdot \phi^* f(m)$. This is equivalent to $\phi(r) f(m) = f(\phi(r)m)$ which follows from the fact that f is S-linear. □

9. (An extra problem whose solution I won't write up) Let $X^2 - aX + b \in \mathbb{R}[X]$ have complex roots $u \pm vi$ with $v \neq 0$. Find a basis of the $\mathbb{R}$-vector space $\mathbb{R}[X]/((X^2 - aX + b)^n)$ with respect to which the linear map "multiplication by X" has matrix

$$\begin{pmatrix} C & I_2 & & \\ & C & I_2 & \\ & & \ddots & I_2 \\ & & & C \end{pmatrix}$$

where $C = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$. This procedure yields the Jordan canonical form for real matrices.