Math 30820 Honors Algebra 4
Homework 5

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Do 4 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

Throughout this problem set R is an **integral domain**, unless otherwise specified.

1. (You must do this problem. It's a more streamlined version of the proof I presented in class.) Suppose K/F is the splitting field of $P(X) \in F[X]$ and $Q(X) \in F[X]$ is an irreducible polynomial with roots α, β.
   (a) Show that $K(\alpha)$ (respectively $K(\beta)$) is the splitting field of $P(X)$ over $F(\alpha)$ (respectively $F(\beta)$).
   (b) Show that $K(\alpha) \cong K(\beta)$.
   (c) Deduce that K/F is a normal extension.
2. (You must do this problem.) Let k be a field and $k(x)$ be the field of rational functions in the variable x. Let $t = \dfrac{P(x)}{Q(x)} \in k(x)$ with P and $Q \neq 0$ coprime in $k[x]$. Denote by $k(t)$ the subextension of $k(x)$ generated by t.
   (a) Show that the polynomial $R(Y) = P(Y) - tQ(Y) \in k(t)[Y]$ is irreducible over $k(t)$ and $R(x) = 0$. [Hint: Use Gauss' lemma and show that $R(Y)$ is irreducible over $k[t, Y]$.]
   (b) Show that the degree of $R(Y)$ as a polynomial in Y is the maximum of the degrees of $P(x)$ and $Q(x)$ as polynomials in x.
   (c) Show that $[k(x) : k(t)] = \max(\deg P(x), \deg Q(x))$.
3. Let K/F and L/F be field extensions. Suppose you are given subextensions $K/K_i/F$ and $L/L_j/F$ for i and j in partially ordered index sets I and J such that if $i \leq i'$ and $j \leq j'$ then $K_i \subset K_{i'}$ and $L_j \subset L_{j'}$. Further assume that I and J satisfy the following property: any two elements of the partially ordered set have an upper bound in the partially ordered set. If $K = \bigcup_{i \in I} K_i$ and $L = \bigcup_{j \in J} L_j$. Show that $KL = \bigcup_{(i,j) \in I \times J} K_i L_j$. [Hint: Show that the RHS is the smallest field that contains K and L.]
4. Show that if K/F and L/F are algebraic extensions then KL/F is also an algebraic extension. [Hint: Use the previous problem and the result for finite extensions in class to show that if $\{u_i\}$ is a basis of K/F and $\{v_j\}$ are a basis of L/F then $\{u_i v_j\}$ span KL/F.]
5. Show that if L/F and K/F are finite extensions such that $[KL : F] = [K : F][L : F]$ then $K \cap L = F$.
6. Artin 15.3.7 on page 473.