Math 30820 Honors Algebra 4
Homework 6

Andrei Jorza

Due Wednesday, 3/1/2017

Do 6 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1. Show that every extension K/F with $[K : F] = 2$ is a normal extension.

2. Let $P \in F[X]$, of degree n, and K be the splitting field of P over F. Show that $[K : F] \mid n!$.

3. Let F be a field, $P \in F[X]$ a *monic* polynomial and K a field that contains all the roots $\alpha_1, \dots, \alpha_n$ of the polynomial $P(X)$, where n is the degree of $P(X)$. The **discriminant** of $P(X)$ is defined as

$$\Delta = \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2$$

   Show that P is separable if and only if $\Delta \neq 0$ and that

$$\Delta = (-1)^{\binom{n}{2}} \prod_{i=1}^{n} P'(\alpha_i)$$

4. (Do one of the 2 parts)
   (a) Consider the polynomial $P(X) = X^5 + pX + q$. Show that it has discriminant

$$\Delta = 5^5 q^4 + 4^4 p^5$$

   (b) (This part is worth 2 extra points) Consider the polynomial $P(X) = X^n + pX + q$. Show that it has discriminant

$$\Delta = (-1)^{\binom{n}{2}} n^n q^{n-1} + (-1)^{\binom{n-1}{2}} (n-1)^{n-1} p^n$$

   [Hint: Use the previous problem.]

5. Let $\alpha \in \mathbb{R}$ such that $\alpha^4 = 5$.
   (a) Is $\mathbb{Q}(i\alpha^2)$ normal over $\mathbb{Q}$?
   (b) Is $\mathbb{Q}(\alpha + i\alpha)$ normal over $\mathbb{Q}(i\alpha^2)$?
   (c) Is $\mathbb{Q}(\alpha + i\alpha)$ normal over $\mathbb{Q}$?

6. Let F be a field of characteristic p that is nor perfect, i.e., the Frobenius homomorphism $\phi : F \to F$ given by $\phi(x) = x^p$ is not surjective. Show that there exist inseparable irreducible polynomials in $F[X]$.

7. Let F be a field of characteristic p and let K/F be a finite extension with $p \nmid [K : F]$. Show that K/F is a separable extension, i.e., for every $\alpha \in K$ the minimal polynomial of α over F is a separable polynomial.

8. Let $F = k(x)$ be the field of rational functions in the variable x with coefficients in some field k. Suppose $\phi : F \to F$ is a field automorphism such that $\phi|_k = \mathrm{id}\,|_k$. Show that there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, k)$ such that $\phi(x) = \dfrac{ax + b}{cx + d}$. [Hint: What is $[F : \operatorname{Im} \phi]$?]