Math 30820 Honors Algebra 4
Homework 7

Andrei Jorza

Due Wednesday, 3/8/2017

Do 6 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

Throughout this problem set $\Phi_n(X)$ is the n-th cyclotomic polynomial.

1. Let p be a prime number. Show that a polynomial $P(X) \in \mathbb{F}_p[X]$ is irreducible if and only if $P(X) \mid X^{p^n} - X$ but $(P(X), X^{p^d} - X) = 1$ for all $d \mid n$, $d < n$.

2. For an integer n write $\omega(n)$ be the number of distinct prime divisors of n, i.e., if $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime factorization then $\omega(n) = k$. Show that

$$\sum_{n \geq 1} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$$

   [Hint: Use the product formula for $\zeta(s)$ from class.]

3. Show that the probability that a monic polynomial of degree n in $\mathbb{F}_p[X]$ is irreducible is $\frac{1}{n} + \varepsilon$ where $|\varepsilon| \leq \frac{1}{p^{n/2}}$.

4. Show that $\Phi_{2n}(X) = \Phi_n(-X)$ for any odd $n > 1$.

5. Let $a \in \mathbb{Z}$. Show that if p is an odd prime divisor of $\Phi_n(a)$ then either $p \mid n$ or $n \mid p - 1$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a field automorphism.
   (a) Show that $f|_{\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}$.
   (b) Show that if $x > 0$ then $f(x) > 0$ and conclude that f is increasing.
   (c) Show that if $|x - y| < \frac{1}{n}$ then $|f(x) - f(y)| < \frac{1}{n}$ and conclude that f is continuous.
   (d) Show that $f = \mathrm{id}_{\mathbb{R}}$.

7. Artin 15.7.5 on page 474.

8. Artin 15.7.12 on page 474.