Math 30820 Honors Algebra 4
Homework 8

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Do 6 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

Throughout this problem set $\Phi_n(X)$ is the n-th cyclotomic polynomial.

1. For a positive integer n we denote by $s(n)$ the largest square-free divisor of n. Show that
$$\Phi_n(X) = \Phi_{s(n)}(X^{n/s(n)})$$
[Hint: Use the Möbius inversion formula.]

2. Show that
$$\Phi_n(1) = \begin{cases} 0 & n = 1 \\ p & n = p^a \\ 1 & n = p_1^{a_1} \cdots p_k^{a_k}, k \geq 2 \end{cases}$$
[Hint: Use induction.]

3. Show that
$$\prod_{1 \leq k \leq n, (k,n)=1} \sin\left(\frac{k\pi}{n}\right) = \frac{\Phi_n(1)}{2^{\varphi(n)}}$$
where φ is Euler's function. Remark that $\Phi_n(1)$ is computed in the previous problem. [Hint: Write $\Phi_n(1)$ as a product over the primitive roots of 1 and use double angle formulas.]

4. Let p be a prime. Let F be the union of the fields of rational functions $\mathbb{F}_p(x) \subset \mathbb{F}_p(\sqrt[p]{x}) \subset \mathbb{F}_p(\sqrt[p^2]{x}) \subset \ldots \subset \mathbb{F}_p(\sqrt[p^n]{x}) \subset \ldots$. Show that F is the smallest perfect field containing $\mathbb{F}_p(x)$.

5. Let p be a prime and $K = \mathbb{Q}(\zeta_p, \sqrt[p]{2})$ be the splitting field of $X^p - 2 \in \mathbb{Q}[X]$. Show that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the subgroup of $\mathrm{GL}(2, \mathbb{F}_p)$ consisting of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. (Recall from last semester that this group, in turn, is of the form $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$.)

6. Let K be the splitting field over $\mathbb{Q}$ of $X^8 - 2$. Show that $\mathrm{Gal}(K, \mathbb{Q}(i)) \cong \mathbb{Z}/8\mathbb{Z}$ and $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{2})) \cong D_8$, the dihedral group with 8 elements.

7. Let $\alpha_1 = \sqrt{1+\sqrt{3}}$, $\alpha_2 = \sqrt{1-\sqrt{3}}$, two roots of the irreducible polynomial $X^4 - 2X^2 - 2 \in \mathbb{Q}[X]$.
   (a) Show that $\mathbb{Q}(\alpha_1) \cap \mathbb{Q}(\alpha_2) = \mathbb{Q}(\sqrt{3})$.
   (b) Show that $\mathbb{Q}(\alpha_1)$, $\mathbb{Q}(\alpha_2)$ and $\mathbb{Q}(\alpha_1, \alpha_2)$ are Galois over $\mathbb{Q}(\sqrt{3})$ and that $\mathrm{Gal}(\mathbb{Q}(\alpha_1, \alpha_2)/\mathbb{Q}(\sqrt{3})) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

8. Show that $K = \mathbb{Q}(\sqrt{2+\sqrt{2}})$ is Galois over $\mathbb{Q}$ and that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$.