Math 30820 Honors Algebra 4
Homework 9

Andrei Jorza

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Do 6 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

Throughout this problem set $\Phi_n(X)$ is the n-th cyclotomic polynomial.

1. Let $p > 2$ be a prime number. Show that $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p}) \subset \mathbb{Q}(\zeta_p)$. [Hint: Compute the discriminant of $X^p - 1$.]

2. Let $P(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree p, a prime number. Suppose $P(X)$ has $p-2$ real roots and one pair of (necessarily conjugate) complex roots. Show that if K is the splitting field over $\mathbb{Q}$ of $P(X)$ then $\mathrm{Gal}(K/\mathbb{Q}) \cong S_p$. You may use the fact that a p-cycle and a transposition generate S_p when p is a prime. For an extra 2 points prove this fact. [Hint: Recall from last semester that (12) and $(12\ldots p)$ generate S_p.]

3. Let K/F be a finite Galois extension and L/F a finite extension such that $K \cap L = F$ and KL/F is Galois. Show that $\mathrm{Gal}(KL/F) \cong \mathrm{Gal}(KL/K) \rtimes_\varphi \mathrm{Gal}(K/F)$ for some homomorphism φ.

4. Let $K = \mathbb{Q}(\zeta_7, \sqrt[7]{2})$ with Galois group $\mathrm{Gal}(K/\mathbb{Q}) \cong \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_7)\}$. Let $H \subset \mathrm{Gal}(K/\mathbb{Q})$ be the subgroup generated by the matrix $A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$. Show that $K^H = \mathbb{Q}(\zeta_7^6 \sqrt[7]{2})$.

5. Let p be a prime and $n \geq 1$. Show that there exists a $K \subset \mathbb{Q}(\zeta_{p^{n+2}})$ such that $K/\mathbb{Q}$ is Galois with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/p^n\mathbb{Z}$.

6. Let $p > 2$ be a prime and $K = \mathbb{Q}(\zeta_p, \sqrt[p]{2})$ the splitting field of $X^p - 2$ over $\mathbb{Q}$. Suppose $K/L/\mathbb{Q}$ is a subextension such that $L/\mathbb{Q}$ is Galois. Show that $L \subset \mathbb{Q}(\zeta_p)$. [Hint: Use the main theorem of Galois theory and your knowledge of normal subgroups of $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$ from last semester.]

7. Let $p > 2$ be a prime and g a generator of the cyclic group $\mathbb{F}_p^\times$. Show that the subextensions $\mathbb{Q}(\zeta_p)/K/\mathbb{Q}$ are all of the form
$$K_d = \mathbb{Q}(\omega_d)$$
as d ranges among the divisors of $p-1$ and $\omega_d = \displaystyle\sum_{i=1}^{(p-1)/d} \zeta_p^{g^{di}}$. [Hint: Use the main theorem of Galois theory.]

8. Let $n \geq 1$.
   (a) Show that $\alpha_n = \zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1} = \sqrt{2 + \sqrt{2 + \sqrt{\cdots + \sqrt{2}}}}$ (n times).
   (b) Show that $\mathrm{Gal}(\mathbb{Q}(\alpha_n)/\mathbb{Q}) \cong \mathbb{Z}/2^n\mathbb{Z}$.