Math 30820 Honors Algebra 4
Homework 10

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Due Wednesday, 4/5/2017

Do 6 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

Throughout this problem set $\Phi_n(X)$ is the n-th cyclotomic polynomial.

1. Suppose K/F is a finite Galois extension and H is a subgroup of $\mathrm{Gal}(K/F)$. Show that $\sigma(K^H) = K^{\sigma H \sigma^{-1}}$.

2. Let $P(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree 4 with roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let $\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $\beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$ and $\beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$. Show that $Q(X) = (X - \beta_1)(X - \beta_2)(X - \beta_3)$ is an irreducible polynomial in $\mathbb{Q}[X]$.

3. Let $p > 2$ be a prime such that $\mathbb{Q}(\zeta_p, \sqrt[p]{2}) \cap \mathbb{Q}(\zeta_p, \sqrt[p]{3}) = \mathbb{Q}(\zeta_p)$. Find a homomorphism $\phi : \mathbb{F}_p^\times \to \mathrm{GL}(2, \mathbb{F}_p) = \mathrm{Aut}(\mathbb{F}_p^2)$ such that

$$\mathrm{Gal}(\mathbb{Q}(\zeta_p, \sqrt[p]{2}, \sqrt[p]{3})/\mathbb{Q}) \cong \mathbb{F}_p^2 \rtimes_\phi \mathbb{F}_p^\times$$

4. Artin 16.1.1 on page 505.

5. Artin 16.5.2 on page 507.

6. Artin 16.9.14 on page 509.

7-8 (This is worth 2 problems) Let p be an odd prime. The point of this exercise is to show that $\mathbb{Q}(\zeta_{p^2}) \cap \mathbb{Q}(\sqrt[p^2]{2}) = \mathbb{Q}$. Write $K = \mathbb{Q}(\zeta_{p^2}) \cap \mathbb{Q}(\sqrt[p^2]{2})$.

   (a) Show that $\mathbb{Q}(\zeta_{p^2}) \cap \mathbb{Q}(\sqrt[p]{2}) = \mathbb{Q}$.

   (b) If $K \neq \mathbb{Q}$ show that $K/\mathbb{Q}$ is Galois of order p.

   (c) Show that $K\mathbb{Q}(\sqrt[p]{2}) = \mathbb{Q}(\sqrt[p^2]{2})$ and deduce that $\mathbb{Q}(\sqrt[p^2]{2})/\mathbb{Q}(\sqrt[p]{2})$ is Galois.

   (d) Show that this extension is never Galois and therefore $K = \mathbb{Q}$.

   (e) Show that $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^2}, \sqrt[p^2]{2})/\mathbb{Q})$ is isomorphic to the matrix group $\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in (\mathbb{Z}/p^2\mathbb{Z})^\times, b \in \mathbb{Z}/p^2\mathbb{Z}\}$.