Math 30820 Honors Algebra 4
Homework 11

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Due Wednesday, 4/12/2017

Do 6 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

Let $C_5 = \mathbb{Z}/5\mathbb{Z}$, D_5 the dihedral group with 10 elements, $F_{20} = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_5)\} \cong \mathbb{Z}/5\mathbb{Z} \rtimes (\mathbb{Z}/5\mathbb{Z})^\times$. You may take for granted that every transitive subgroup of S_5 is isomorphic to one of the groups C_5, D_5, F_{20}, A_5 and S_5.

1. Let G be a group, F a field, and $\chi_1, \dots, \chi_n : G \to F^\times$ distinct group homomorphisms. Show that the functions χ_i are F-linearly independent, i.e., if $a_1, \dots, a_n \in F$ and $g \mapsto a_1\chi_1(g)+a_2\chi_2(g)+\cdots+a_n\chi_n(g)$ is the 0 function then all the coefficients a_i are 0. [Hint: Choose a minimal linear dependence and mimick the proof of the fact that a minimal spanning set for a vector space is a basis.]

2. Let $P(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree n and K its splitting field over $\mathbb{Q}$. Fixing an ordering of the roots of $P(X)$ consider $\mathrm{Gal}(K/\mathbb{Q})$ as a subgroup of S_n. Let $H < S_n$ be a subgroup and write $S_n/H = \{\sigma_1 H, \dots, \sigma_d H\}$. Suppose there exists $\theta \in K$ such that $h(\theta) = \theta$ for all $h \in H$ and $\{\sigma_1(\theta), \dots, \sigma_d(\theta)\}$ are all distinct. Show that $R(X) = \displaystyle\prod_{i=1}^{d}(X - \sigma_i(\theta))$ is a separable polynomial in $\mathbb{Q}[X]$.

3. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $P(X) = X^5 - 5X + 12 \in \mathbb{Q}[X]$.
   (a) Show that $P(X)$ is irreducible.
   (b) Show that $10 \mid [K : \mathbb{Q}]$. [Hint: $P(X)$ has two pairs of complex conjugate roots.]
   (c) Assume that $P(X)$ is solvable by radicals (it is). Show that $\mathrm{Gal}(K/\mathbb{Q}) \cong D_5$, the dihedral group with 10 elements. [Hint: What is the discriminant of $P(X)$?]

4. Artin 16.9.12 on page 509.

5. Artin 16.9.13 on page 509.

6. Artin 16.9.18 on page 509.

7. Artin 16.M.7 part (b) on page 512. (Part (a) we did in class.)

8. Let k be a field and t an indeterminate. Recall from a previous homework that we have identified $\mathrm{Aut}(k(t)/k)$ with the group $\mathrm{PGL}(2, k)$ via fractional linear transformations. Suppose $H < \mathrm{PGL}(2, k)$ is a subgroup of order n.
   (a) Let $P_H(X) = \displaystyle\prod_{h \in H}(X - h(t))$. Show that $P_H(X) \in k(t)^H[X]$.
   (b) Show that $k(t)^H$ is generated over k by the coefficients of $P_H(X)$.

(c) Suppose $k = \mathbb{F}_2$. Show that

$$\mathbb{F}_2(t)^{\mathrm{Aut}(\mathbb{F}_2(t)/\mathbb{F}_2)} = \mathbb{F}_2\left(\frac{(t^2+t+1)^3}{t^2(t+1)^2}\right)$$

[Hint: Recall from last semester that $\mathrm{PGL}(2, \mathbb{F}_2) = \mathrm{GL}(2, \mathbb{F}_2) \cong S_3$. You don't need the computationally intensive part (b), although it would lead to the same answer..]