Math 30820 Honors Algebra 4
Homework 12

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Do 8 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1. Consider the variables $x_1, \dots, x_n, t$ and the Taylor expansion
$$\phi_t(x) = \frac{xt}{1-e^{-xt}} = 1 + \frac{x}{2}t + \frac{x^2}{12}t^2 - \frac{x^4}{720}t^4 + O(t^6)$$
Let $c_1 = x_1 + \cdots + x_n$, $c_2 = \sum x_i x_j$, $\dots$, $c_n = x_1 \cdots x_n$ be the elementary symmetric polynomials. Show that
$$\phi_t(x_1)\phi_t(x_2)\cdots\phi_t(x_n) = 1 + \frac{c_1}{2}t + \frac{c_1^2+c_2}{12}t^2 + \frac{c_1c_2}{24}t^3 + \frac{-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4}{720}t^4 + O(t^5)$$
[Hint: This is really just a question about symmetric polynomials written in terms of elementary symmetric polynomials.] (The LHS is referred to as the Todd class while the c_i are referred to as Chern classes.)

2. Let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{Q}[X]$ be any monic polynomial. Show that for every $\varepsilon > 0$ there exists a polynomial $Q(X) = X^n + b_{n-1}X^{n-1} + \cdots + b_1X + b_0 \in \mathbb{Q}[X]$ such that (a) $|b_k - a_k| < \varepsilon$ for all k and (b) $Q(X)$ is irreducible in $\mathbb{Q}[X]$. [Hint: You can produce $Q(X)$ that satisfies the hypotheses of the Eisenstein irreducibility criterion for a ring of the form $R = \frac{1}{N}\mathbb{Z}$ where N is large enough.]

3. Let p be a prime. Show that there exists a monic irreducible polynomial $P(X)$ of degree p with 2 complex conjugate roots are $p-2$ real roots. [Hint: Use the previous problem.]

4. Show that if G is any finite group there exist finite extensions $L/K/\mathbb{Q}$ such that L/K is Galois with $\mathrm{Gal}(L/K) \cong G$. [Hint: Embed G into S_p for some prime p and find L as the splitting field of a degree p irreducible polynomial with exactly two complex roots. Use the previous exercise.]

5. Artin 16.12.6 on page 511.

6. Artin 16.12.7 on page 511.

7. Artin 16.M.10 on page 512.

8-10 (Worth 3 problems) Artin 16.M.11 on page 512. As stated part (c) is incorrect (in fact the example from class $X^4 + 5X + 5$ gives a contradiction as $\delta = 55\sqrt{5}$ and $\gamma = -\sqrt{5}$ so $\gamma\delta = -5^2 \cdot 11$). Prove instead the following version of (c): Show that $\gamma\delta$ and $\gamma\varepsilon$ are in F.