Math 30820 Honors Algebra 4
Homework 13

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Do 4 of the following questions. Some questions may be obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1. Let K/F be a finite Galois extension, $\sigma \in \mathrm{Gal}(K/F)$ and $x \in K^\times$.
   (a) Show that $N_{K/F}(x/\sigma(x)) = 1$.
   (b) Show that $\mathrm{Tr}_{K/F}(x - \sigma(x)) = 0$.

2. Suppose K/F is a finite Galois extension with $\mathrm{Gal}(K/F)$ cyclic of order n generated by an automorphism σ. Show that if $\alpha \in K$ has the property $\mathrm{Tr}_{K/F}(\alpha) = 0$ then there exists $\beta \in K$ such that $\alpha = \beta - \sigma(\beta)$. [Hint: Similarly to the theorem from class, look at

$$\frac{1}{\mathrm{Tr}_{K/F}(\theta)} \sum_{i=1}^{n-1} \sigma^i(\theta) \sum_{j=0}^{i-1} \sigma^j(\alpha)$$

   for suitably chosen θ.]

3. Let $a, b \in \mathbb{Z}$. Find all $(x, y) \in \mathbb{Q}^2$ such that $x^2 + axy + by^2 = 1$.

4. (Correction of Artin 16.M.11.(c)) Keep the notations from 16.M.11. Prove that if $\gamma \neq 0$ then $\delta\gamma \in F$ if and only if $G = C_4$. Similarly, prove that if $\varepsilon \neq 0$ then $\delta\varepsilon \in F$ if and only if $G = C_4$.

5-6 (Worth 2 problems) Let $L/K/F$ be finite extensions such that L/F is Galois. Write $G = \mathrm{Gal}(L/F)$ and $H = \mathrm{Gal}(L/K)$ and let $\{\sigma_1, \ldots, \sigma_n\}$ be a complete set of representatives in G of G/H, i.e., $G/H = \{\sigma_1 H, \ldots, \sigma_n H\}$. For $\alpha \in K$ define

$$\mathcal{P}_{\alpha,L}(X) = \prod_{\sigma \in \mathrm{Gal}(L/F)/\mathrm{Gal}(L/K)} (X - \sigma(\alpha)) = \prod_{i=1}^{n} (X - \sigma_i(\alpha))$$

   (a) Show that $\mathcal{P}_{\alpha,L}(X)$ is a well-defined polynomial with coefficients in F. (Careful: K/F need not be Galois.)
   (b) Define $\mathrm{Tr}_{K/F,L}(\alpha)$ and $N_{K/F,L}(\alpha)$ as the coefficients of $\mathcal{P}_{\alpha,L}(X)$ as follows:

$$\mathcal{P}_{\alpha,L}(X) = X^n - \mathrm{Tr}_{K/F,L}(\alpha) X^{n-1} + \cdots + (-1)^n N_{K/F,L}(\alpha)$$

   Show that $\mathrm{Tr}_{K/F,L} : K \to F$ is a homomorphism of F-vector spaces and $N_{K/F,L} : K^\times \to F^\times$ is a group homomorphism.
   (c) If L'/L is a finite extension such that L'/F is Galois show that $\mathcal{P}_{\alpha,L}(X) = \mathcal{P}_{\alpha,L'}(X)$.
   (d) Deduce that $\mathcal{P}_{\alpha,L}(X)$ (and therefore also $\mathrm{Tr}_{K/F,L}$ and $N_{K/F,L}$) does not depend on the choice of Galois extension L/F. (We can therefore drop the subscript L from notation to obtain trace $\mathrm{Tr}_{K/F}$ and norm $N_{K/F}$ for all finite extensions.)