

# Math 40520 Theory of Number

## Homework 2

Due Friday, 9/14, in class

**Do 5 of the following 7 problems.**

1. Show that the equation

$$x^2 + y^2 + z^2 = 20152015$$

has no integral solutions. [Hint: Try congruences modulo powers of 2.]

2. Show that the equation  $7x^2 - y^2 = 2^z$  has no integer solutions.

3. Show that the equation  $x^2 - xy + y^2 = 2018$  has no integer solutions.

4. Solve explicitly the equation  $251x + 613y = 838$  in the integers.

5. Let  $a_1, \dots, a_n$  be integers. Show that there exist integers  $u_1, \dots, u_n$  such that

$$a_1u_1 + \dots + a_nu_n = \gcd(a_1, a_2, \dots, a_n).$$

6. Consider the diophantine equation

$$2x^2 + 7y^2 = 1$$

(a) Show that it has no integral solutions but that it has  $(1/3, 1/3)$  as a rational solution.

(b) Suppose  $n \geq 2$  is an integer not divisible by 3. Show that there exist integers  $x, y$  such that

$$2x^2 + 7y^2 \equiv 1 \pmod{n}$$

[Hint: Use the rational solution from above.]

7. Let  $p$  be a prime and consider the rational number

$$\frac{m}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p-1}$$

If  $p > 2$  show that  $p \mid m$ . [Hint: consider the function  $f : \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times$  defined by  $f(x) = x^{-1}$ .]