Math 40520 Theory of Number
Homework 8

Due Friday, 11/9, in class

Do 5 of the following problems.

1. Let a be a nonzero integer.
   (a) Show that there exists at least one prime p such that $\left(\frac{a}{p}\right) = 1$. [Hint: You seek a prime p such that $a \equiv b^2 \pmod{p}$ for some integer b.]
   (b) Show that there are infinitely many primes p such that $\left(\frac{a}{p}\right) = 1$. [Hint: We did this in class for $a = -1$ and $a = -3$.]
2. Let a be an odd integer and $n \geq 3$ be an integer. Show that a is a square modulo 2^n if and only if $a \equiv 1 \pmod{8}$. [Hint: In class we showed that 17 is a square mod 2^n and indeed $17 \equiv 1 \pmod{8}$.]
3. Let $p \equiv 3 \pmod{4}$ be a prime. Compute $\displaystyle\sum_{x=0}^{p-1} \left(\frac{x^4 - 1}{p}\right)$.
4. Consider the prime $p = 97$. Compute $\displaystyle\sum_{x=0}^{p-1} \left(\frac{x^3 - 1}{p}\right)$.
5. Let p be an odd prime. Compute $\displaystyle\sum_{0 \leq x < y \leq p-1} \left(\frac{xy}{p}\right)$.
6. Compute, using quadratic reciprocity, the Legendre symbol $\left(\frac{-123}{2017}\right)$.
7. Exercise 4.3 on page 89 in the textbook.
8. Exercise 4.7 on page 90 in the textbook.