Math 43900 Fall 2017 Problem Solving
Lecture 1, Aug. 21

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The many stages of problem solving

1. I know this by heart. E.g., $(\ln x)' = \dfrac{1}{x}$.
2. I've seen problems exactly like this. E.g., $\displaystyle\int \frac{dx}{\sqrt{x^2 - x + 15}}$.
3. I've seen many time the methods necessary to do this problem. E.g., compute $(x^{x^x})'$.
4. Perhaps I can reinterpret/simplify the problem to something I've seen before. Questions: do I have to technical comfort to do this? Or the inventivity?
5. This seems like a geometry problem but it might really be something else.
6. What idea/trick/perspective could the author of the problem have been aiming for? Remember: this problem has been done by someone in a reasonable amount of time. It's not a research problem. This can be extremely helpful.
7. "How do I" questions. How do I choose a problem to work on? How do I brainstorm for (or build a repository of) helpful ideas? What do I do when I get stuck?
8. I don't even understand what the problem is saying.
9. Uncontrolled laughter at the absurdity and impossibility of the problem.

Useful tips

1. Work out some small examples.
2. Work under simplifying assumptions.
3. Is this problem similar to anything else you've encountered?
4. What do you know about the setup of the problem from previous experience?
5. If you have any intuition about the problem, can you articulate it?
6. If you formulate some observations from small cases can you try and guess a strategy?
7. Can you outright guess the answer?
8. What could the problem writer have been thinking about? Any clues in the statement of the problem?

Exercises

1. Find the positive integers n such that $n^2 - 10n - 22$ equals the product of the digits of n written in base 10. (IMO 1968)

2. You write n real numbers on a circle with the following property: whenever you see x, y, z consecutively written (in clockwise direction) then $xy + 1 = z$. What can n be? (IMO 2018)

3. Two players players take turns placing chess knights on a chessboard, the first player using white knights, the second player using black knights. There are only two rules:

   (a) A player must place a knight on an unoccupied square only.

   (b) No two white knights may attack each other.

   What's the largest number of white knights the first player can guarantee to play no matter what the second player does? (IMO 2018)

4. Define the polynomials $P_n(X)$ for $n \geq 0$ by $P_0(X) = 1$, $P_n(0) = 0$ for $n \geq 1$ and

$$P'_{n+1}(X) = (n+1)P_n(X+1).$$

   Factor $P_{100}(1)$.