Math 80220 Algebrai Number Theory
Problem Set 1

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Due Friday, Feb 2

Throughout this problem set you will use traces and norms. Recall from that that $\mathrm{Tr}_{L/K}(x) = \sum \sigma(x)$ and $N_{L/K}(x) = \prod \sigma(x)$ where $\sigma : L \to \overline{K}$ are the embeddings of L fixing K. When $L = K(\alpha)$ one may enumerate these embeddings by the roots β of the minimal polynomial of α over K, sending α to β and fixing K.

Do 2 of the following problems.

1. Let m, n be coprime square-free integers which are not 1. Let $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$.

   (a) Suppose $m \equiv 2, n \equiv 3 \pmod 4$. Use traces to show that elements of $\mathcal{O}_K$ are of the form
   $$\frac{a + b\sqrt{m} + c\sqrt{n} + d\sqrt{mn}}{2}$$
   where $a, b, c, d \in \mathbb{Z}$. Use norms to $\mathbb{Q}(\sqrt{n})$ to show that a, c are even while b and d have the same parity and conclude that
   $$\mathcal{O}_K = \mathbb{Z}[\sqrt{m}, \sqrt{n}, \frac{\sqrt{m} + \sqrt{mn}}{2}]$$

   (b) Suppose $m, n \equiv 1 \pmod 4$. Again use traces to show that elements of $\mathcal{O}_K$ are of the form
   $$\alpha = \frac{a + b\sqrt{m} + c\sqrt{n} + d\sqrt{mn}}{4}$$
   where $a, b, c, d \in \mathbb{Z}$ are all of the same parity. Show that
   $$\beta = \frac{1 + \sqrt{m} + \sqrt{n} + \sqrt{mn}}{4} \in \mathcal{O}_K$$
   and that one may write
   $$\frac{r + s\sqrt{m} + t\sqrt{n}}{2} = \alpha - d\beta$$
   with $r, s, t \in \mathbb{Z}$ such that $r + s + t$ even (recall that $(1 + \sqrt{m})/2$ and $(1 + \sqrt{n})/2$ are algebraic integers). Conclude that
   $$\mathcal{O}_K = \mathbb{Z}[\frac{1 + \sqrt{m}}{2}, \frac{1 + \sqrt{n}}{2}, \frac{1 + \sqrt{m} + \sqrt{n} + \sqrt{mn}}{4}]$$

2. Let $m \equiv 1 \pmod 9$ be a square-free integer $\neq 1$ and let $K = \mathbb{Q}(\sqrt[3]{m})$.

   (a) Show that $\mathrm{Tr}_{K/\mathbb{Q}}(a + b\sqrt[3]{m} + c\sqrt[3]{m^2}) = 3a$ and $N_{K/\mathbb{Q}}(a + b\sqrt[3]{m} + c\sqrt[3]{m^2}) = a^3 + b^3 m + c^3 m^2 - 3abcm$.

   (b) Show that
   $$\delta = \frac{1 + \sqrt[3]{m} + \sqrt[3]{m^2}}{3} \in \mathcal{O}_K$$

(c) Use traces to show that elements of $\mathcal{O}_K$ are of the form

$$\alpha = \frac{am + b\sqrt[3]{m} + c\sqrt[3]{m}}{3m}$$

with $a, b, c \in \mathbb{Z}$. Use norms to show that $m \mid b, c$ and so we may in fact write

$$\alpha = \frac{r + s\sqrt[3]{m} + t\sqrt[3]{m^2}}{3}$$

for integers r, s, t.

(d) Use the fact that $\alpha - t\delta \in \mathcal{O}_K$ to conclude that

$$\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{m}, \frac{1 + \sqrt[3]{m} + \sqrt[3]{m^2}}{3}]$$

3. Let $M/L/K$ be finite extensions. Let $\alpha_1, \ldots, \alpha_n$ be a K-basis of L and $\beta_1, \ldots, \beta_m$ be an L-basis of M. Recall from algebra that $\{\alpha_i\beta_j\}$ is a K-basis of M. Show that

$$\operatorname{disc}_{M/K}((\alpha_i\beta_j)_{1\leq i\leq n, 1\leq j\leq m}) = \operatorname{disc}_{L/K}((\alpha_i))^{[M:L]} N_{L/K} \operatorname{disc}_{M/L}((\beta_j)).$$