Math 80220 Algebraic Number Theory
Problem Set 3

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due Friday, February 16

1. Show that the ideal $I = (x^2, y)$ of $\mathbb{C}[x, y]$ divides the product of prime ideals $(x)(x, y)$ but I is not a product of prime ideals.

2. Show that
$$\int \cdots \int_{\substack{x_1,\ldots,x_a,y_1,\ldots,y_b \geq 0 \\ \sum x_i + \sum y_j \leq t}} \prod y_j \prod dx_i \prod dy_j = \frac{t^{a+2b}}{(a+2b)!},$$
and deduce that the volume of the region
$$\{(x_1, \ldots, x_r, y_1, z_1, \ldots, y_s, z_s) \in \mathbb{R}^n \mid |x_1| + \cdots + |x_r| + 2\sqrt{y_1^2 + z_1^2} + \cdots + 2\sqrt{y_s^2 + z_s^2} \leq t\}$$
with respect to the usual volume form on $\mathbb{R}^n$ is $\dfrac{2^{r-s}\pi^s t^n}{n!}$.

3. Compute the discriminant of the number field $\mathbb{Q}(\zeta_p)$.

4. Let K be a number field. Show that $\mu_\infty(K) = \{z \in K \mid z^n = 1 \text{ for some } n \geq 1\}$ is a finite set of the form μ_m for some even $m \geq 2$.

5. Let K be a number field. Show that $\mathcal{O}_K^\times = \{\alpha \in K \mid |N_{K/\mathbb{Q}}(\alpha)| = 1\}$.