Math 80220 Algebraic Number Theory
Problem Set 4

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due Friday, February 23

1. Let $f(X) = X^3 - 3X + 1$.

   (a) Show that $f(X)$ is irreducible over $\mathbb{Q}$ and has 3 real roots. Let $K = \mathbb{Q}(\alpha)$ where α is a root. Show that
   $$3^4\mathcal{O}_K \subset \mathbb{Z}[\alpha] \subset \mathcal{O}_K.$$
   [Hint: What is the discriminant of $1, \alpha, \alpha^2$?]

   (b) Show that $\alpha, \alpha+2$ are units and that $(\alpha+1)^3 = 3\alpha(\alpha+2)$ and conclude that $(3)\mathcal{O}_K = (\alpha+1)^3$ is the factorization of (3) as a product of prime ideals in $\mathcal{O}_K$.

   (c) Show that $\mathcal{O}_K = \mathbb{Z}[\alpha] + (3)\mathcal{O}_K$. [Hint: Compute the cardinality of $\mathcal{O}_K/(3)$ to show that the equality occurs after quotienting by (3).]

   (d) Deduce that $\mathcal{O}_K = \mathbb{Z}[\alpha]$. [Hint: if e_1, e_2, e_3 is an integral basis for $\mathcal{O}_K$ what can you say about the highest power of 3 in the denominators of e_i?]

   (e) Show that $\mathcal{O}_K/(2) \cong \mathbb{F}_{2^3}$ and deduce that K has class number 1.

   (f) What is the subgroup $\mu_K \subset \mathcal{O}_K^\times$ of roots of unity? [Hint: What is the degree of ζ_n over $\mathbb{Q}$?]

   (g) Show that α and $\alpha+2$ are independent in $\mathcal{O}_K^\times$. Are they a basis for the free part of $\mathcal{O}_K^\times$? [Hint: For the first part, show that the three real roots of $f(X)$ lie in the intervals $(-2,-1)$, $(0,1)$ and $(1,2)$ and if α and $\alpha+2$ have a dependence then the same is true for the other two roots. For the second part compute $1/(\alpha+2)$.]

2. Let $K = \mathbb{Q}(\sqrt[3]{7})$ with $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{7}]$.

   (a) Show that $\mathrm{Cl}(K) \cong \mathbb{Z}/3\mathbb{Z}$ generated by $(2, \sqrt[3]{7}+1)$. [Feel free to use a computer for multiplying fractional ideals.]

   (b) Show that $2 - \sqrt[3]{7}$ is a unit.

   (c) Show that in fact $2 - \sqrt[3]{7}$ generates the free part of $\mathcal{O}_K^\times$:

      i. Suppose $u > 1$ is a generator for the rank 1 abelian group $\mathcal{O}_K^\times$. Let $\sigma(u) = re^{i\theta}$ and $\overline{\sigma}(u)$ be the two complex conjugates of u. Show that $u = r^{-2}$.

      ii. Show that
      $$\mathrm{disc}(1, u, u^2) = -4\sin^2(\theta)(r^3 + r^{-3} - 2\cos(\theta))^2$$
      and deduce that
      $$|\,\mathrm{disc}(u)| < 4(u^3 + u^{-3} + 6)$$
      [Hint: For fixed $c = \cos(\theta)$ maximize $(1-c^2)(x-2c)^2 - x^2$ where $x = r^3 + r^{-3}$.]

      iii. Show that $u^3 > |\,\mathrm{disc}(K)|/4 - 7$. Show that $\mathrm{disc}(K) = -1323$ and deduce that $u^3 > 323.75$. Show that $2 - \sqrt[3]{7} = u^{-k}$ for some $k > 0$ and deduce that $2 - \sqrt[3]{7}$ is a generator of the free part of $\mathcal{O}_K^\times$. [Feel free to use a calculator for the numerical estimates.]