Math 80220 Algebraic Number Theory
Problem Set 5

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due Friday, March 2

1. Let K be a number field, $\mathfrak{p}$ a nonzero prime ideal, and $v = v_{\mathfrak{p}}$ the associated non-archimedean place. The completion K_v is a topological space with topology defined by its metric space structure. Show that K_v is a topological field, i.e., that addition and multiplication : $K_v \times K_v \to K_v$ and inversion $K_v^\times \to K_v^\times$ are continuous functions where $K_v \times K_v$ is endowed with the product topology and $K_v^\times$ with the subset topology.

2. Show that $\mathbb{A}_K = \prod'_{\{\mathcal{O}_v\}} K_v$ is a locally compact topological ring.

3. Recall from class the multiplicative map $|\cdot|_{\mathbb{A}_K} : \mathbb{A}_K \to [0, \infty)$ defined by
$$|(x_v)|_{\mathbb{A}_K} = \prod_v |x_v|_v.$$
Show that if $x \in K^\times$ then $|\iota(x)|_{\mathbb{A}_K} = 1$ where $\iota : K \hookrightarrow \mathbb{A}_K$ is given by $\iota(x) = (x)_v$.

4. If G is a locally compact group there is a unique (up to scalars) measure vol_G on G with the following properties: $\mathrm{vol}_G(X) < \infty$ for all compact subsets $X \subset G$ and $\mathrm{vol}(gX) = \mathrm{vol}(X)$ for all $g \in G$. Such a measure is called a (left) Haar measure on G. Let vol_v be a Haar measure on K_V and vol_K be a Haar measure on $\mathbb{A}_K$. Show that:
   (a) if $X \subset K_v$ is compact and $a \in K_v$ then $\mathrm{vol}_v(aX) = |a|_v \, \mathrm{vol}_v(X)$.
   (b) if $X \subset \mathbb{A}_K$ is compact and $a \in \mathbb{A}_K$ then $\mathrm{vol}_K(aX) = |a|_{\mathbb{A}_K} \, \mathrm{vol}_K(X)$.