Math 80220 Algebraic Number Theory
Problem Set 6

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due Friday, March 9

1. Let S be a finite set of finite places of a number field K and $\mathcal{O}_S = \{x \in K \mid v(x) \geq 0, \forall v \notin S\}$ be the subring of S-integers. Show that $\mathcal{O}_S^\times$ is a finitely generated abelian group of rank $\operatorname{rk} \mathcal{O}_K^\times + |S|$.

2. Let v be any place of a number field K. For $w \neq v$ fix constants $\delta_w > 0$ such that $\delta_w = 1$ for almost all w. Show that there exists $x \in K^\times$ such that $|x|_w \leq \delta_w$ for all $w \neq v$. [Hint: Use adelic Minkowski.]

3. Let v be any place of a number field K. Write $\mathbb{A}_K^v = \prod'_{w \neq v, \{\mathcal{O}_w\}} K_w$ for the topological subring of $\prod_{w \neq v} K_w$ consisting of tuples $(x_w)_{w \neq v}$ such that $x_w \in \mathcal{O}_w$ for almost all w. Show that while $K \subset \mathbb{A}_K$ is discrete, $K \subset \mathbb{A}_K^v$ is dense.

4. Find an element $x \in \mathbb{Q}$ such that $|x - 1|_7, |x - 2|_{11}, |x + 2004|_\infty < 1/10$.

5. Let K be a number field.

   (a) Show that if I is an ideal there exists a number field L/K such that $I\mathcal{O}_L$ is principal. [Hint: some power of I must be principal.]

   (b) Show that there exists a number field L/K such that every ideal of $\mathcal{O}_K$ becomes principal in $\mathcal{O}_L$.