Math 80220 Algebraic Number Theory
Problem Set 10

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Algebraic K-theory is an important homological algebra theory. Borel proved that if F is a number field and $n \geq 0$ then $K_n\mathcal{O}_F$ is a finitely generated abelian group. In fact $K_0\mathcal{O}_F \cong \mathrm{Cl}(F) \oplus \mathbb{Z}$ has rank 1, $K_1\mathcal{O}_F \cong \mathcal{O}_F^\times$ has rank $r_1 + r_2 - 1$, and for $n \geq 2$:

$$\operatorname{rk} K_n\mathcal{O}_F = \begin{cases} 0 & n = 2i \\ r_1 + r_2 & n = 4i+1 \\ r_2 & n = 4i-1 \end{cases}$$

where r_1 is the number of real places and r_2 is the number of complex places. When S is a finite set of finite places then $K_0\mathcal{O}_{F,S} \cong \mathrm{Cl}_S(F) \oplus \mathbb{Z}$ has rank 1, $K_1\mathcal{O}_{F,S} \cong \mathcal{O}_{F,S}^\times$ has rank $r_1 + r_2 + |S| - 1$, and for $n \geq 2$, $\operatorname{rk} K_n\mathcal{O}_{F,S} = \operatorname{rk} K_n\mathcal{O}_F$.

1. Let F be a number field and S a finite (possibly empty) set of prime ideals of F. Let

$$\zeta_{F,S}(s) = \sum_{I \text{ ideal of } \mathcal{O}_{F,S}} \frac{1}{||I||^s} = \prod_{\mathfrak{p} \in S} \left(1 - \frac{1}{||\mathfrak{p}||^s}\right) \zeta_F(s).$$

Show that for $m \geq 1$ the order of vanishing at $s = 1 - m$ of the S-Dedekind zeta function is

$$\operatorname{ord}_{s=1-m} \zeta_{F,S}(s) = \operatorname{rk} K_{2m-1}\mathcal{O}_{F,S}.$$

[Hint: This is straightforward using the functional equation.] (This is an elementary example of the Beilinson conjecture.)

2. Consider the functions $\mathcal{B}(x) = \dfrac{x}{e^x - 1}$ and $\mathcal{B}(x, z) = \dfrac{xe^{xz}}{e^x - 1}$.

   (a) Show that the Taylor expansion of $\mathcal{B}(x)$ around $x = 0$ is of the form

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \frac{x^n}{n!}$$

   where $B_n \in \mathbb{Q}$ are such that $B_{2k+1} = 0$ if $k \geq 1$. The coefficients B_n are known as the **Bernoulli numbers**. [Hint: Show that $\mathcal{B}(x) + x/2$ is even.]

   (b) Show that "$B^n = (B+1)^n$" is satisfied by the Bernoulli numbers in the sense that

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$$

   when $n \neq 1$. [Hint: Compute the power series product $\mathcal{B}(x)\exp(x)$.]

(c) Show that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$ and $B_4 = -\frac{1}{30}$. [Hint: work $\pmod{x^2}$ and then use the previous part.]

(d) Show that

$$\mathcal{B}(x,z) = \sum_{n\geq 0} B_n(z)\frac{x^n}{n!}$$

where $B_n(z) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} z^k$ are polynomials in z of degree n (called the Bernoulli polynomials).

(e) With B_n and $B_n(z)$ as above, show that for $n, m \geq 1$ (with the convention $0^0 = 1$) one has

$$\begin{aligned}\sum_{k=0}^{n-1} k^{m-1} &= \frac{B_m(n) - B_m}{m} \\ &= \frac{1}{m}\sum_{k=0}^{m-1}\binom{m}{k} B_k n^{m-k}\end{aligned}$$

[Hint: Take $\sum_{m\geq 1}(\cdot)\frac{x^m}{m!}$ of the above expressions and compare generating functions.]

3. Take for granted that

$$\sin(z) = z\prod_{n\geq 1}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$

(a) Show that

$$z\,\mathrm{cotan}(z) = 1 + 2\sum_{n=1}^{\infty}\frac{z^2}{z^2 - n^2\pi^2}$$

[Hint: Take logarithmic derivative of $\sin(z)$.]

(b) Show that

$$z\,\mathrm{cotan}(z) = 1 + \sum_{n\geq 2}(2i)^n B_n \frac{z^n}{n!}$$

[Hint: Plug in $x = 2iz$ in $\mathcal{B}(x)$ and recall that $e^{iz} = \cos(z) + i\sin(z)$.]

(c) Show that

$$\frac{z^2}{z^2 - n^2\pi^2} = -\sum_{k\geq 1}\frac{z^{2k}}{n^{2k}\pi^{2k}}$$

and conclude that

$$z\,\mathrm{cotan}(z) = 1 - 2\sum_{k\geq 1}\frac{z^{2k}\zeta(2k)}{\pi^{2k}}$$

(d) Deduce that for $n \geq 1$,

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}$$

(e) Show that for $n \geq 1$

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}$$

and thus make sense of the famous expression

$$1 + 2 + 3 + \cdots = -\frac{1}{12}$$

[Hint: Use the functional equation.]