Math 30810 Honors Algebra 3
Homework 3

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Due Wednesday, September 25

Do 5 questions.

1. Artin 2.4.3
2. Artin 2.4.7
3. Artin 2.6.2
4. Let G be a group with subgroups H and K. Show that $H \cup K$ is a group if and only if one of H and K contains the other.
5. Show that every cyclic group is abelian.
6. Prove that every subgroup of a cyclic group is cyclic.
7. Let G be a group and H a subgroup. Show that the function $f : G \to G$ defined by $f(x) = x^{-1}$ gives a bijection between G/H and $H\backslash G$.
8. For a matrix $A \in M_{n\times n}(\mathbb{C})$ define $e^A = I_n + A + A^2/2! + A^3/3! + \cdots$. You may assume that this expression always converges to a matrix $e^A \in M_{n\times n}(\mathbb{C})$. If $S \in \mathrm{GL}_n(\mathbb{C})$ show that $e^{SAS^{-1}} = Se^AS^{-1}$.
9. In the context of the previous exercise show that if A is upper triangular with $a_1, \ldots, a_n$ on the diagonal then e^A is upper triangular with $e^{a_1}, \ldots, e^{a_n}$ on the diagonal and conclude that $\det(e^A) = e^{\mathrm{Tr}(A)}$. As an optional exercise show that for any matrix A, $\det(e^A) = e^{\mathrm{Tr}(A)}$ and deduce that e^A is always invertible. [Hint: You may use the following standard fact from linear algebra, that for every matrix A you can find an invertible matrix S such that SAS^{-1} is upper triangular.]