Math 30810 Honors Algebra 3
Midterm 1

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Due Wednesday, October 9

Do 5. Please do not collaborate with anyone else. You may use the textbook and your notes. Do not use Internet resources. You may use any exercise on this set to prove any other exercise, even without proof.

1. Let G be a group and $g \in G$. Suppose $g^m = e$ and $g^n = e$ where m and n are coprime integers. Show that $g = e$.

2. Let G be a group.

   (a) Assume that H and K are subgroups and $|H| = |K| = p$ is a prime number. Show that either $H = K$ or $H \cap K = \{e\}$.

   (b) Let G be a group and $H_1, \ldots, H_k$ be distinct subgroups of G. Suppose that each group H_i has order p, a fixed prime number. Show that $H_1 \cup \ldots \cup H_k$ has exactly $(p-1)k + 1$ elements.

3. Suppose G is a finite group and p is a prime number such that every element $g \in G - \{e\}$ has order p. Show that $p - 1 \mid |G| - 1$. [Hint: use exercise 2.]

4. Let $G = \mathrm{GL}_2(\mathbb{R})$ and B the subgroup of upper triangular matrices. Show that

$$\mathrm{GL}_2(\mathbb{R}) = B \sqcup B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B.$$

   Define the relation $\sim$ on invertible matrices by saying that $X \sim Y$ if and only if there exist $U, V \in B$ such that $A = UXV$. Show that $\sim$ is an equivalence relation and conclude that $\mathrm{GL}_2(\mathbb{R})/\sim$ has two elements with representatives $\{I_2, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\}$.

5. Artin 2.8.6 on page 73.

6. Artin 2.8.8 on page 73.

7. Artin 2.8.10 on page 73.

8. Suppose G is a group and $x, y \in G$. If $\mathrm{ord}(x)$ and $\mathrm{ord}(y)$ are coprime show that

$$\mathrm{ord}(xy) = \mathrm{ord}(x)\,\mathrm{ord}(y).$$

9. For a prime p and an integer n we denote $v_p(n)$ the power of p in the factorization of n. E.g., $v_3(12) = 1$, $v_2(5/4) = -2$, etc. Suppose $a \equiv b \pmod p$ are two integers. Show that

$$v_p(a^n - b^n) = v_p(a - b) + v_p(n).$$

   Conclude that $\mathrm{ord}(p + 1 \mod p^n) = p^{n-1}$ for every odd prime p but $\mathrm{ord}(3 \mod 2^n) = 2^{n-2}$.