Math 30810 Honors Algebra 3
Homework 8

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Due in class Wednesday, November 6

Do 5.

1. Let G be a group, H a subgroup of G and N a normal subgroup of G. Show that $HN = NH$ is a subgroup of G as well.

2. A **short exact sequence** of groups is a sequence of group homomorphisms

$$1 \to N \xrightarrow{f} G \xrightarrow{g} K \to 1$$

such that $f : N \to G$ is injective, $g : G \to K$ is surjective, and $\operatorname{Im} f = \ker g$. A **section** of such an exact sequence is defined to be a group homomorphism $s : K \to G$ such that $g \circ s = \mathrm{id}_K$.

   (a) Show that in the short exact sequence above $N \cong f(N) \lhd G$ and $G/f(N) \cong K$.

   (b) Suppose that the exact sequence above admits a section $s : K \to G$. Show that for every $x \in G$ one can find $n \in N$ such that $x = f(n)s(g(x))$ and deduce that $G \cong N \rtimes K$ is a semidirect product.

   (c) (Extra credit) Show that if $G \cong N \rtimes K$ then one can find an exact sequence $1 \to N \to G \to K \to 1$ that admits a section $s : K \to G$.

3. Let $\zeta = e^{2\pi i/3}$, $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. Let $G = \langle x, y \rangle \subset \mathrm{GL}(2, \mathbb{C})$ be the subgroups generated by x and y.

   (a) Show that x has order 4, y has order 3 and $xy = y^2 x$.

   (b) Show that G has order 12 with $G = \{y^b x^a \mid 0 \le a < 4, 0 \le b < 3\}$.

   (c) Show that $G \cong \mathbb{Z}/3\mathbb{Z} \rtimes_\phi \mathbb{Z}/4\mathbb{Z}$ for some $\phi : \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$. [Hint: Use the criterion for when a group is a semidirect product.]

4. Consider the homomorphism $\phi : S_3 \to \mathrm{Inn}(S_3) = \mathrm{Aut}(S_3)$ defined by $\phi_g(x) = gxg^{-1}$. Consider the groups $G_0 = S_3 \rtimes_\phi S_3$, $G_1 = S_3 \rtimes_\phi A_3$ and $G_2 = A_3 \rtimes_\phi A_3$.

   (a) Show that $G_2 \cong (\mathbb{Z}/3\mathbb{Z})^2$ [Hint: $A_3 \cong \mathbb{Z}/3\mathbb{Z}$],

   (b) Show that $G_2 \lhd G_1$ with $G_1/G_2 \cong \mathbb{Z}/2\mathbb{Z}$,

   (c) Show that $G_1 \lhd G_0$ with $G_0/G_1 \cong \mathbb{Z}/2\mathbb{Z}$.

   [Hint: To show normality you can use a criterion from a previous problem set. No need to do any conjugation.]

5. Show that S_n is generated by the transpositions (12), (23), …, $(n-1, n)$. [Hint: $(23)(12)(23) = (13)$. Recall that in class we showed that S_n is generated by all transpositions.]

6. (a) Show that $(12\dots n)(i, i+1)(12\dots n)^{-1} = (i+1, i+2)$ for $i+2 \leq n$.
   (b) Show that $(12\dots n)^k(12)(12\dots n)^{-k} = (k+1, k+2)$ for $k+2 \leq n$.
   (c) Deduce that S_n is generated by (12) and $(12\dots n)$. [Hint: Use the previous problem.]

7-8 (This counts as two problems) Let $n \geq 3$ be an odd number. Consider the group homomorphism $\phi : (\mathbb{Z}/n\mathbb{Z})^\times \to \operatorname{Aut}(\mathbb{Z}/n\mathbb{Z})$ given by $\phi_a(x) = ax$. Recall that the dihedral group $D_{2n} = \{F^u R^v \mid 0 \leq u \leq 1, 0 \leq v < n\}$.

   (a) For $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ and $b \in \mathbb{Z}/n\mathbb{Z}$ define $\Psi_{a,b}(F^u R^v) := (FR^b)^u (R^a)^v$. Show that $\Psi_{a,b} \in \operatorname{Aut}(D_{2n})$. [Hint: Use the previous problem.]
   (b) Show that $\Psi : (\mathbb{Z}/n\mathbb{Z}) \rtimes_\phi (\mathbb{Z}/n\mathbb{Z})^\times \to \operatorname{Aut}(D_{2n})$ is an injective group homomorphism.
   (c) Show that Ψ is surjective, i.e., that every automorphism of D_{2n} is of the form $\Psi_{a,b}$ for some a and b and conclude that

$$\operatorname{Aut}(D_{2n}) \cong (\mathbb{Z}/n\mathbb{Z}) \rtimes_\phi (\mathbb{Z}/n\mathbb{Z})^\times$$

   [Hint: Use part (a).]
   (d) (Extra credit) For a group G the group of outer automorphisms is defined as $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$, a group since $\operatorname{Inn}(G) \triangleleft \operatorname{Aut}(G)$ from a previous homework. Show that $\operatorname{Out}(D_{2n}) \cong (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}$.

Remark: If G is a group and $\phi : \operatorname{Aut}(G) \to \operatorname{Aut}(G)$ is the identity homomorphism then the semidirect product $G \rtimes_\phi \operatorname{Aut}(G)$ is called the holomorph of G. The point of this exercise was to show that $\operatorname{Aut}(D_{2n})$ when $n \geq 3$ was odd was the holomorph of $\mathbb{Z}/n\mathbb{Z}$. In fact this is true for all n.