Math 30810 Honors Algebra 3
Homework 11

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Due in class Wednesday, December 4

Do 5.

1. Let G be a group. If $g, h \in G$ define their commutator as $[g, h] = ghg^{-1}h^{-1}$. The commutator subgroup $[G, G] < G$ is the group generated by all $\{[g, h] \mid g, h \in G\}$.
   (a) Show that $a[b, c]a^{-1} \in [G, G]$ for all $a, b, c \in G$.
   (b) Show that $[G, G] \lhd G$ and $G^{\mathrm{ab}} = G/[G, G]$ is an abelian group. The group G^{ab} is called the abelianization of the group G.
   (c) Show that if A is an abelian group then $\mathrm{Hom}(G, A) = \mathrm{Hom}(G^{\mathrm{ab}}, A)$. (This is taken to mean that G^{ab} is the largest abelian quotient of G.

2. See Problem 1 for the definition of abelianization. You do not need to solve that problem.
   (a) Determine S_n^{ab} for all n.
   (b) (See Problem 1) Let p be a prime and G a nonabelian group of order p^3. Show that $[G, G] = Z(G)$.

3. Let $n \geq 5$ and H a subgroup of S_n. Assume that H is not A_n or S_n. Show that $[S_n : H] \geq n$. [Hint: As in a previous problem set, if H is a subgroup of G then G acts by left multiplication on G/H giving a homomorphism $G \to S_{G/H}$.]

4. (a) Suppose G is a group and $g, h \in G$. Show that gh and hg are conjugate.
   (b) A permutation $\sigma \in S_3$ is said to be good if for every group G and every elements $g_1, g_2, g_3 \in G$, the two products $g_1g_2g_3$ and $g_{\sigma(1)}g_{\sigma(2)}g_{\sigma(3)}$ are conjugate in G. Show that σ is good if and only if $\sigma \in \langle(123)\rangle$. [Hint: conjugate matrices have the same trace.]

5. Artin 7.M.12 on page 228.

6. Let R be a ring. An idempotent element of R is an element $e \in R$ such that $e^2 = e$. Consider $eR = \{ex \mid x \in R\}$. Show that $(eR, +_R, \cdot_R, 0_R, e)$ is a ring.

7. Show that $\mathbb{C}[x][[y]] \neq \mathbb{C}[[y]][x]$.

8. Artin 11.1.6 on page 354.

9. Artin 11.1.7 on page 354.