Math 30810 Honors Algebra 3
Second Exam

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Do 5.

1. Suppose m and n are integers such that $m > n!$. Let G be a **simple** group of order m which acts on a set of cardinality n. Show that G acts trivially, i.e., $gx = x$ for all g and x. [Hint: Recall the two equivalent definitions of group action.]

2. Let $(G, \cdot)$ be a group, $(A, +)$ an abelian group, and $H = \mathrm{Maps}(G \to A)$.
   (a) Show that H is a group under addition of functions with identity element given by the 0 function.
   (b) Show that G acts on H via $(g\phi)(x) = \phi(xg)$.
   (c) Show that the orbit of a group homomorphism $\phi \in H$ under the action of G is in bijection with $\mathrm{Im}\,\phi \subset A$.

3. Consider the homomorphism $\phi : \mathbb{Z}/6\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$ defined by $a \mapsto \phi_a$ where $\phi_a \in \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$ is defined as $\phi_a(x) = 2^a x$. In the group $G = \mathbb{Z}/5\mathbb{Z} \rtimes_\phi \mathbb{Z}/6\mathbb{Z}$ write $R = (0_{\mathbb{Z}/6\mathbb{Z}}, 1_{\mathbb{Z}/5\mathbb{Z}})$ and $F = (1_{\mathbb{Z}/6\mathbb{Z}}, 0_{\mathbb{Z}/5\mathbb{Z}})$. Compute the order of R, F and write FRF^{-1} in the form $F^x R^y$ for some explicit integers x, y.

4. A finite group G acts on a finite set S. For $g \in G$ we denote $S^g = \{s \in S \mid gs = s\}$.
   (a) Show that if g and h are conjugate in G then there is a bijection between S^g and S^h.
   (b) Show that the number of orbits of G acting on S is equal to
   $$|G|^{-1} \sum_{\text{conjugacy classes } C_g} |C_g||S^g|$$
   where the sum is taken over the distinct conjugacy classes of G.

   [Hint: Recall from the homework Burnside's formula: if a finite group G acts on a finite set S then the number of orbits of G acting on S is $|G|^{-1} \sum_{g \in G} |S^g|$.]

5. For an abelian group A (written additively) we denote $A[2] = \{a \in A \mid 2a = 0\}$. Show that for all $n \geq 2$ we have $\mathrm{Hom}(S_n, A) \cong A[2]$.

6. What is the order of the automorphism group $\mathrm{Aut}(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z})$?

7. Let $n \geq 3$ be odd. Find all conjugacy classes in the dihedral group D_n of order $2n$.

8. (a) Show that (123) and (132) are not conjugate in A_3 or A_4.
   (b) Show that if $n \geq 5$ is odd then $(12\ldots, n-2, n-1, n)$ and $(12\ldots n-2, n, n-1)$ are not conjugate in A_n.

9. Let G be a group such that $G/Z(G)$ is a cyclic group. Show that G is abelian.