Math 40520 Theory of Number
Homework 6

Due Wednesday, 9/30, in class

Do 5.

1. Compute $\binom{194871}{1610}$ mod 385. [Hint: Use our theorem for binomial coefficients modulo primes (Lucas' theorem) and the Chinese Remainder Theorem.]

2. (This is a more sophisticated looking, yet easier, version of the previous problem.) A *Sophie-Germaine* prime is a prime p such that $q = 2p + 1$ is also a prime (conjecturally there are infinitely many such primes, the largest known having about 388k digits). Suppose $p \geq 7$ is a Sophie-Germaine prime and $q = 2p + 1$. Show that
$$\binom{pq + pq^2}{pq} \equiv 30q - 2p \equiv 58p + 30 \pmod{pq}$$
[Hint: Same as for the previous problem, but it's easier to write down the digits in bases p and q.]

3. Find all solutions of the equation $x^3 - x - 1 \equiv 0 \pmod{5^k}$ for $k = 1, 2, 3$.

4. Solve $x^{11} \equiv 7 \pmod{32}$. (You have two means of solving this: either primitive roots, or Hensel's lemma.)

5. Let $p > 3$ be a prime number. Find a solution in $\mathbb{Z}_{p^6}$ to the equation
$$x^3 \equiv 1 + p^2 \pmod{p^6}$$

6. Find all solutions of the congruence $x^3 + 4x^2 + 19x + 1 \equiv 0 \pmod{147}$. [Hint: $147 = 3 \cdot 7^2$.]

7. Determine $v_7(\varphi(200!))$.

8. Show that there exists no positive integer n such that $\varphi(n) = 62$.