Math 30820 Honors Algebra 4
Homework 2

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Due Wednesday, 2/5/2020

Do 5.

1. Let R be a PID and $\operatorname{Tor}_R(M) = \{m \in M \mid \exists a \in R - \{0\}, am = 0\}$.
   (a) Show that $\operatorname{Tor}_R(M)$ is an R-submodule of M.
   (b) If $f : M \to N$ is a homomorphism of R-modules then $f(\operatorname{Tor}_R(M)) \subset \operatorname{Tor}_R(N)$.
   (c) Show that $M/\operatorname{Tor}_R(M)$ is free.

2. Let F be a field and $A \in M_{n\times n}(F)$ be a matrix. Recall that in class we defined the following $F[X]$-module M_A: as an abelian group $M_A = F^n$ and scalar multiplication is given by $P(X) \cdot v := P(A)v$ where $P(A) \in M_{n\times n}(F)$ and F^n is interpreted as $M_{n\times 1}(F)$. Suppose $S \in \mathrm{GL}(n, F)$. Show that $M_A \cong M_{SAS^{-1}}$ as $F[X]$-modules.

3. Consider the ring $R = F[X]/(X^2)$. Suppose M is a finitely generated R-module. Show that there exists some matrix $A \in M_{n\times n}(F)$ for some $n \geq 1$, with $A^2 = 0$, such that $M \cong M_A$ from the previous exercise. [Hint: Show that M is naturally a finite dimensional vector space over F and define A as the matrix of multiplication by X.]

4. Show that a finitely generated abelian group is a free abelian group if and only if it contains no elements of finite order other that the identity element.

5. Artin 14.1.4 on page 437.

6. Artin 14.7.7 on page 439.

7. Artin 14.7.8 on page 439.

8. Suppose R is a commutative ring in which the "diagonalization lemma" holds, i.e., for any homomorphism $f : R^n \to R^m$ of free R-modules of finite rank, there exists a basis of R^n and a basis of R^m with respect to which the matrix of f is a diagonal matrix possibly padded with extra rows and columns. Show that every finitely generated ideal of R is principal. (Such rings are called Bézout domains.)

9. Let R be a PID. Suppose $a_1 \mid \ldots \mid a_m$ and $b_1 \mid \ldots \mid b_n$ are nonzero elements of R such that
$$R/(a_1) \oplus \cdots \oplus R/(a_m) \cong R/(b_1) \oplus \cdots \oplus R/(b_n).$$
   Show that $m = n$ and $(a_i) = (b_i)$ for all i.