Math 30820 Honors Algebra 4
Homework 3

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Do 5.

Throughout this problem set R is an **integral domain**, unless otherwise specified.

1. Show that the ideal $I = (2, X)$ of $R = \mathbb{Z}[X]$ is not free as an R-module and determine its rank, i.e., the cardinality of a maximal linearly independent subset of I.
2. Let R be a ring, I an ideal of R and M an R-module.
   (a) Show that $IM = \{\sum_{\text{finite}} a_i m_i \mid a_i \in I, m_i \in M\}$ is an R-submodule of M.
   (b) Show that M/IM is an R/I-module.
3. Consider $\mathbb{C}$ as a $\mathbb{Z}[i]$-module under usual multiplication of complex numbers. Determine the torsion submodule of the $\mathbb{Z}[i]$-module $\mathbb{C}/\mathbb{Z}[i]$.
4. Consider the ring $A = \mathbb{F}_2[X]/(X^2 - X)$ with 4 elements. Show that torsion elements of the free A-module A do not form a submodule. (Note that A is not a domain so this does not contradict the statement from class.)
5. Let M be a finitely generated R-module and $\{m_1, \ldots, m_n\}$ a linearly independent subset of M.
   (a) Show that $N = Rm_1 + \cdots + Rm_n$ is free $\cong R^n$.
   (b) If $\{m_1, \ldots, m_n\}$ is a maximal linearly independent subset show that M/N is torsion, i.e., every element of M/N is annihilated by a nonzero element of R.
6. Consider the R-module $M = R^n \oplus N$ where N is a torsion module, i.e., $N = \mathrm{Tor}(N)$. Let $e_1, \ldots, e_n$ be the standard basis of R^n and $t_1, \ldots, t_n \in N$ arbitrary elements. Show that $v_1 = e_1 + t_1, \ldots, v_n = e_n + t_n$ are linearly independent and the map $f : M \to M$ defined as the identity on N and sending $e_i \mapsto v_i$ is an isomorphism. (The point of this exercise is that while $N = \mathrm{Tor}(M)$ is well-defined solely in terms of M, the free part R^n is not as every basis can be changed by torsion elements to get another basis.)
7. Let $\phi : R \to S$ be a ring homomorphism and M an S-module. For $r \in R$ and $m \in M$ define $r \cdot m := \phi(r)m$, the later being scalar multiplication in M by $\phi(r) \in S$.
   (a) Show that this operation yields an R-module structure on the abelian group M. Call $\phi^* M$ this R-module.
   (b) If $f : M \to N$ is a homomorphism of S-modules define $\phi^* f : \phi^* M \to \phi^* N$ by $\phi^* f(m) = f(m)$. Show that $\phi^* f \in \mathrm{Hom}_R(\phi^* M, \phi^* N)$.

8. Let $X^2 - aX + b \in \mathbb{R}[X]$ have complex roots $u \pm vi$ with $v \neq 0$. Find a basis of the $\mathbb{R}$-vector space $\mathbb{R}[X]/((X^2 - aX + b)^n)$ with respect to which the linear map "multiplication by X" has matrix

$$\begin{pmatrix} C & I_2 & & \\ & C & I_2 & \\ & & \ddots & I_2 \\ & & & C \end{pmatrix}$$

where $C = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$. This procedure yields the Jordan canonical form for real matrices.

9. For an ideal I of a domain R we denote $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some integer } n \geq 0\}$.

   (a) Show that $\sqrt{I}$ is an ideal of R.

   (b) If J is an ideal of $R[X]$ and $\mathscr{L}(J)$ is the ideal of leading coefficients of polynomials in J show that $\mathscr{L}(\sqrt{J}) \subset \sqrt{\mathscr{L}(J)}$.